18 March 2009
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Wall-crossing in moduli spaces of Higgs bundles

(w/ D. Gaiotto, G. Moore)

Greg's talk:
Physical context for KS WCF

$$d=4 \xrightarrow{S^1} d=3 \qquad \begin{array}{c} \mathcal{M} \\ \downarrow \\ \mathcal{B} \end{array}$$

$\mathcal{X}_\gamma : \mathcal{M} \to \mathbb{C}^\times$

M-theory in 11-d spacetime $\mathbb{R}^3 \times T^*C \times \mathbb{R}^{1,2} \times S^1_R$ ($C \subset T^*C$)

N M5-brane on $\{pt\} \times C \times \mathbb{R}^{1,2} \times S^1_R$

(2,0) theory on $S^1 \to$ 5d SYM on $C \times \mathbb{R}^{1,2}$

BPS vacua in the 3d theory $\leftrightarrow$ ~~Higgs bundle~~ Solution of Hitchin's eq. on C

$U(N)$-connection D on a bundle V over C

$\varphi \in \Omega^{1,0}(\mathrm{End}\, V)$

Hitchin eq: $\bar{\partial}_D \varphi = 0$

$$R^2 [\varphi, \bar{\varphi}] = F_D.$$

2

$$\begin{tikzcd}[column sep=small]
 & (2,0) \text{ theory in 6 dimensions} \arrow[dl, "C"'] \arrow[dr, "T^2"] & \\
\mathcal{N}=2 \text{ theory in 4 dimensions} \arrow[d, "S^1"] & & \mathcal{N}=4 \text{ SYM in 4 dimensions} \arrow[dl, "C"] \\
\mathcal{M};\ \mathcal{N}=4 \text{ theory in 3 dimensions} \arrow[d, "S^1"] & & \\
\sigma\text{-model in 2 dimensions on } \mathcal{M} & &
\end{tikzcd}$$

$\mathcal{N}=(4,4)$

$$\begin{array}{ccc} \mathcal{M} & & (E,\varphi) \\ \downarrow & & \mapsto \\ \mathcal{B} & & \text{char poly } \varphi \end{array}$$

Spectral curve:

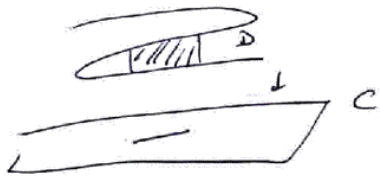
$$S = \{(z,x) \mid \det(x-\varphi(z))=0\} \subset T^*C$$

We consider (tamely) ramified Higgs bundles on C

(φ, A) have poles at $z_i \in C$

BPS states:

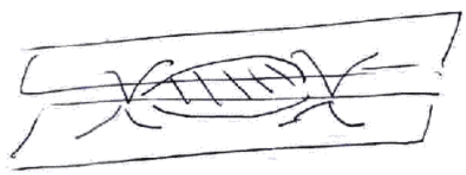
Open M2-branes ending on the spectral curve $S \subset T^*C$

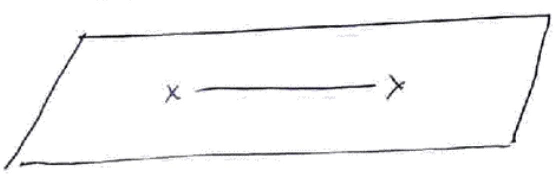


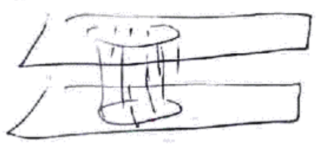
Mass of BPS states $M = |Z_\gamma|$ where

$$Z_\gamma = \oint_\gamma \lambda \qquad \lambda = p\,dx$$

$$\partial D = \gamma. \qquad \gamma \in H_1(S, \mathbb{Z})$$







4

$$\mathcal{X}_\gamma : \mathcal{M} \times \underset{\substack{\cup \\ \zeta}}{\mathbb{C}^\times} \longrightarrow \mathbb{C}^\times$$

Jumps: Define $\mathcal{K}_\gamma$ to be the transformation

$$\mathcal{X}_{\gamma'} \to \mathcal{X}_{\gamma'} (1 + \mathcal{X}_\gamma)^{\langle \gamma, \gamma' \rangle}$$

Then the collection $\{\mathcal{X}_\gamma\}$ jumps by the symplectomorphism

$\mathcal{K}_\gamma^{\Omega(\gamma; u)}$ as ζ crosses the ray $\ell_\gamma = \{\zeta : Z(\gamma;u)/\zeta \in \mathbb{R}_-\}$

Asymptotics: $\lim_{\zeta \to 0} \mathcal{X}_\gamma \exp(-\pi R Z(\gamma; u)/\zeta)$ exists

What is $\mathcal{M}_\zeta$ at $\zeta \in \mathbb{C}^\times$?

Consider the complex connection

$$\nabla_z = R\zeta^{-1} \varphi_z + D_z$$
$$\nabla_{\bar z} = R\zeta \bar\varphi_{\bar z} + D_{\bar z}$$

Hitchin eq $\Rightarrow$ ∇ is flat.

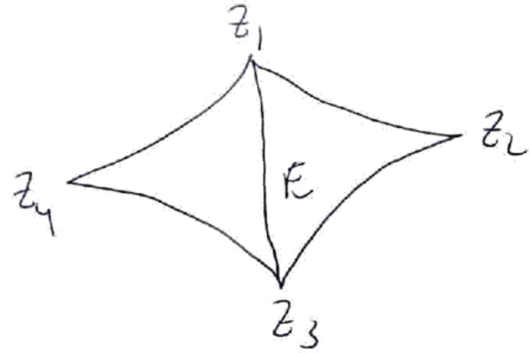
From now on,
Specialize to $G = SU(2)$

Fock-Goncharov coordinates

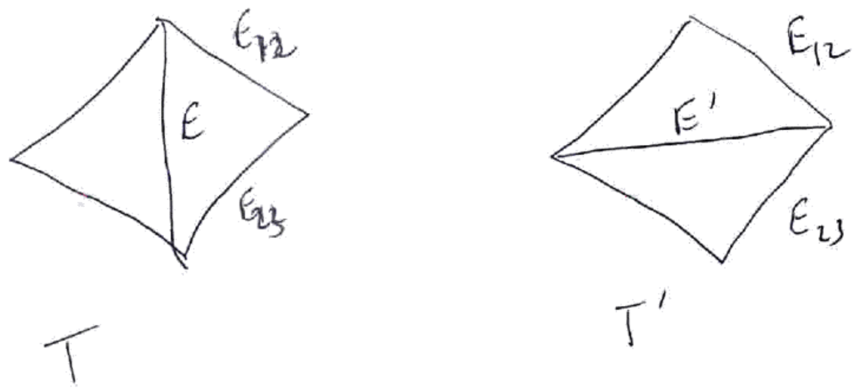
Choose one of the two monodromy eigenspaces at each singular point z_i, + monodromy eigenvectors s_i.

5

Choose a triangulation T of C s.t. the vertices = the singularities.
We construct a function χ_E^T for each edge T.



$$\chi_E^T = -\frac{(s_1 \wedge s_2)(s_3 \wedge s_4)}{(s_2 \wedge s_3)(s_4 \wedge s_1)}$$



$$\chi_{E'}^{T'} = (\chi_E^T)^{-1} \qquad \chi_{E_{12}}^{T'} = \chi_{E_{12}}^T (1 + \chi_E^T)$$

$$\chi_{E_{23}}^{T'} = \chi_{E_{23}}^T (1 + (\chi_E^T)^{-1})^{-1}$$

6

How to go from $\{X^T_E\}$ to desired X_γ ?

We'll construct a triangulation $T(\zeta, u)$ (canonical)

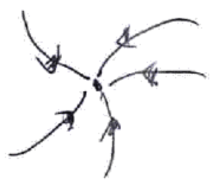
and we'll construct a mapping $E \mapsto \gamma_E \in H_1(S_u, \partial)$

WKB foliation of C

Consider the 1-form $\lambda = x\,dz$ on S

Look at curves on S with $\langle \partial_t, \lambda \rangle \in \mathbb{R}_+ \zeta$

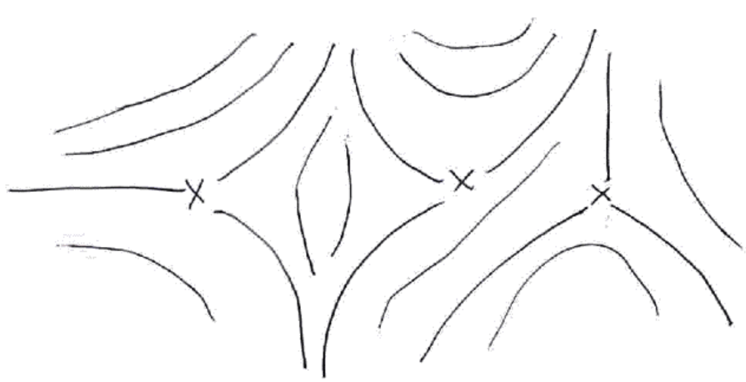
Local behavior (on C):



near poles

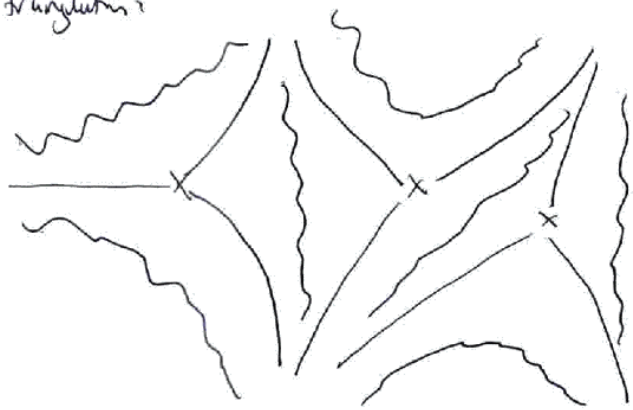
near zeros

$\mathrm{Tr}\,\varphi^2$

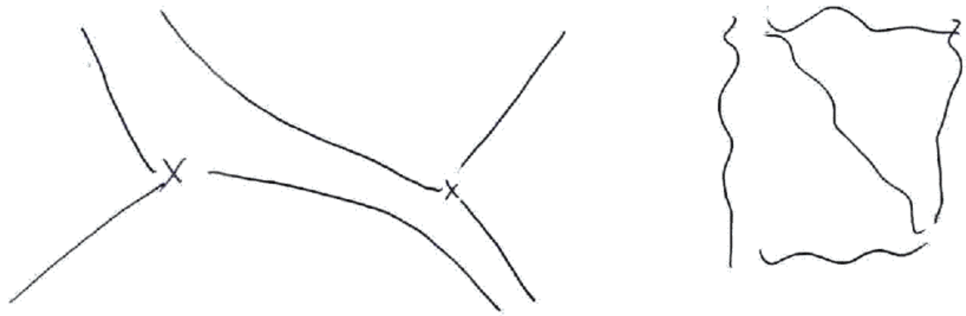


7

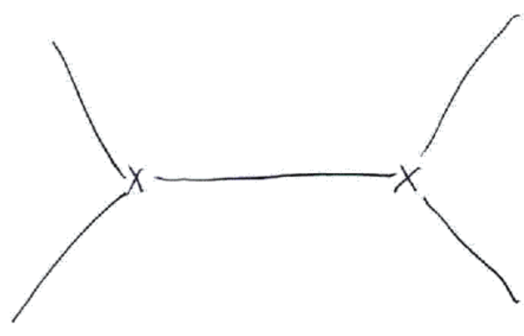
choose one generic WKB curve from each cell – the edges of the triangulation:



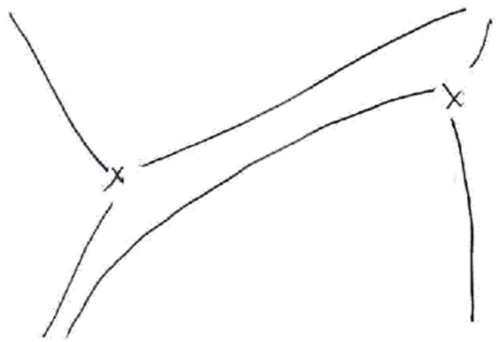
How can it change?

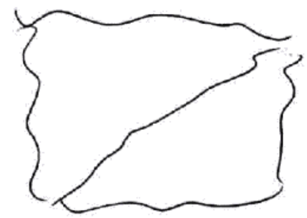


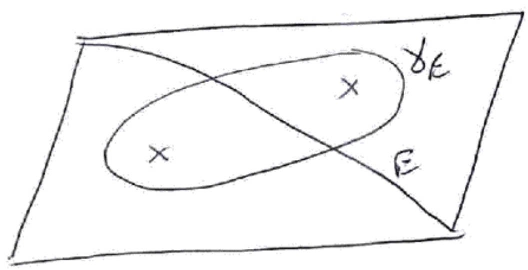
critical value of ϑ:

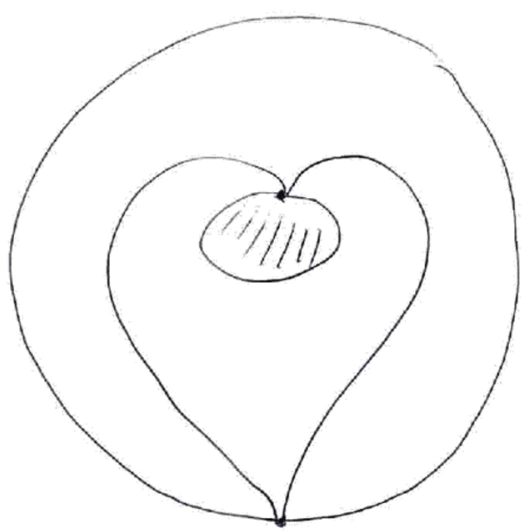


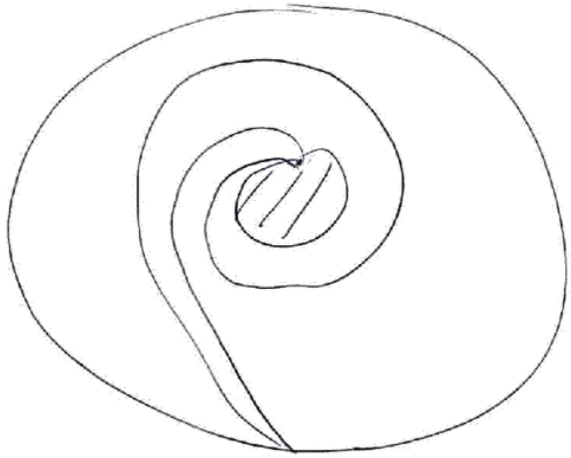
vary ϑ further:






γ_E
E





$$\{X_\gamma^-\}$$

$$\updownarrow \quad \mathcal{K}_\gamma^{-2}$$

$$\{X_\gamma^+\}$$