Tetrad Approaches to Numerical Relativity

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References

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Advantages of tetrad versus metric

- The tetrad metric, and therefore the raising and lowering of indices, is trivial. There are many fewer terms in the equations, with a more consistent structure.
- Fewer connection coefficients (24 versus 40 in 4D, 9 versus 18 in 3D).
- Variables are more closely related to physical/geometric quantities.
- Tetrad gauge conditions (the evolution of the acceleration and angular velocity of the tetrad frames) can be formulated in a coordinate-independent way.
- Most of the variables are coordinate scalars, so Lie derivatives in time derivative operators do not involve derivatives of the shift vector.

Potential disadvantages of tetrad formalism

- Need gauge conditions to evolve *both* tetrad frames and coordinates.
- Frames may not have any fixed orientation relative to the constant time hypersurfaces on which the numerical evolution is defined, so the “spatial” tetrad directional derivatives may contain coordinate time derivatives and also may not coincide with spatial coordinate axes.
- Commutation of tetrad directional derivatives is non-trivial.
- There are more tetrad vector components to evolve than coordinate metric components (9 versus 6 in 3D).
- May be harder to develop asymptotically stationary gauge conditions.

Threading approaches (1 + 3)

Tetrad congruence world lines not forced to be orthogonal to constant-time hypersurfaces. We consider three types of tetrad evolution gauges consistent with symmetric hyperbolicity of a “Christoffel” system.

- Fixed acceleration and angular velocity of tetrad frames.
- “Lorentz” gauge of van Putten and Eardley.
- “Nester” gauge of Estabrook, et al.

Hypersurface-orthogonal frames (3 + 1)

Tetrad congruence worldlines are forced to be orthogonal to the constant-time hypersurfaces. A dynamic equation for the lapse evolves the hypersurfaces and the tetrad acceleration. I consider two basic types of evolution systems:

- An “Einstein-Christoffel” system, based on first-order equations connecting the connection coefficients.
- An “Einstein-Bianchi” system (Choquet-Bruhat and York) which uses the Bianchi identities to evolve the Riemann tensor.

Both systems are simplified to use the minimum number of variables, while maintaining symmetric hyperbolicity.

Reduction to practice

- Relate tetrad variables and directional derivatives to coordinates and coordinate derivatives required for numerical calculation. Derive eigenvectors of characteristic matrix for arbitrary directions of propagation.

1D test applications:

- colliding plane waves, with general polarization
- spherical symmetry (Schwarzschild geometry)

Notation

$\alpha,\beta,\gamma,...$	space-time tetrad indices (0-3)
$\lambda,\mu,\nu,...$	space-time coordinate indices (0-3)
$a,b,c,...$	spatial tetrad indices (1-3)
$i,j,k,...$	spatial coordinate indices (1-3)

Tetrad vectors λ_α^μ, $\lambda_\alpha \bullet \lambda_\beta = \eta_{\alpha\beta}$

Tetrad directional derivatives $D_\alpha = \lambda_\alpha^\mu \frac{\partial}{\partial x^\mu}$

Coordinate metric $g^{\mu\nu} = \eta^{\alpha\beta}\lambda_\alpha^\mu \lambda_\beta^\nu$

Coordinates

Introduce a *tetrad* lapse and shift such that

$$D_0 = \frac{1}{\alpha}\left(\frac{\partial}{\partial t} - \beta^k \frac{\partial}{\partial x^k}\right).$$

Project the spatial tetrad vectors into the hypersurface along the tetrad world lines to define a truly spatial directional derivative operator $\partial_a = e_a^k \frac{\partial}{\partial x^k}$ such that $D_a = A_a D_0 + \partial_a$.

A_a is the 3-velocity of the tetrad worldline relative to the hypersurface normal. Necessary conditions for non-singular coordinates are 1) $\alpha > 0$ and 2) $A_a A_a < 1$. The inverse spatial metric $h^{\mu\nu} = e_a^\mu e_a^\nu$ should be positive-definite.

Connection coefficients

$$\Gamma_{\alpha\beta\gamma} = \lambda_\alpha \bullet \nabla_\gamma \lambda_\beta = -\Gamma_{\beta\alpha\gamma}$$
$$= \frac{1}{2}\lambda_\beta \bullet \left[\lambda_\alpha, \lambda_\gamma\right] + \frac{1}{2}\lambda_\gamma \bullet \left[\lambda_\alpha, \lambda_\beta\right] - \frac{1}{2}\lambda_\alpha \bullet \left[\lambda_\beta, \lambda_\gamma\right]$$

Space-time split in E-W dyadic notation

- Acceleration 3-vector $a_b = \Gamma_{b00} = \lambda_0 \bullet \left[\lambda_b, \lambda_0\right]$
- Angular velocity 3-vector $\omega_b = -\frac{1}{2}\varepsilon_{abc}\Gamma_{bc0}$
- "Extrinsic curvature" $K_{ab} = \Gamma_{b0a}$
- Spatial connection dyadic $N_{ab} = \frac{1}{2}\varepsilon_{bcd}\Gamma_{cda}$

Twist vector, vanishes iff tetrad is hypersurface orthogonal, antisymmetric part of K_{ab}:

$$\Omega_a \equiv \frac{1}{2}\varepsilon_{abc}K_{bc} \ .$$

Antisymmetric part of N_{ab}, only part which transforms non-trivially under conformal rescalings of spatial metric:

$$n_a \equiv \frac{1}{2}\varepsilon_{abc}N_{bc} \ .$$

Evolution of the (in general) 9 K_{ab} and 9 N_{ab} are determined by the Einstein equations and ordering identities, while the evolution of the 3 a_b and the 3 ω_b is determined by gauge conditions.

Einstein equations

Riemann tensor $R_{\alpha\beta\gamma\delta} = D_\gamma \Gamma_{\alpha\beta\delta} - D_\delta \Gamma_{\alpha\beta\gamma} + ...$

Initial value equations

$$G_{00} = R_{2323} + R_{3131} + R_{1212} = D_b\left(2n_b\right) + ...$$

$$G_{01} = R_{0212} + R_{0313} = -D_2K_{12} - D_3K_{13} + D_1\left(K_{22} + K_{33}\right) + ...$$

Evolution equations

$$\begin{aligned} R_{ba} - \delta_{ba}G_{00} &= -R_{0b0a} + R_{cbca} - \delta_{ba}\left(R_{2323} + R_{3131} + R_{1212}\right) \\ &= D_0K_{ab} - D_a a_b - \varepsilon_{acd}D_cN_{db} + ... \end{aligned}$$

Note that form of the the evolution equations is not symmetric, even though the Ricci tensor is symmetric.

Identities for evolution and constraints

Riemann identities

Ordering identities $\left(R_{\alpha\beta\gamma\delta}=R_{\gamma\delta\alpha\beta}\right)$:

$$D_0 N_{ab}+D_a \omega_b+\varepsilon_{acd} D_c K_{db}+\ldots=0 \ ,$$

$$-D_2 N_{12}-D_3 N_{13}+D_1\left(N_{22}+N_{33}\right)+\ldots=0 \ .$$

Cyclic identity:

$$R_{0123}+R_{0231}+R_{0312}=D_b\left(2\Omega_b\right)+\ldots=0 \ .$$

Also require gauge evolution equations for a_b and ω_b.
Simplest (but not suitable in general): a_b and ω_b are fixed functions of the coordinates (fixed gauge).

Gauge Evolution

Lorentz gauge (van Putten and Eardley)

Connection 1-forms $\omega_{\mu\alpha\beta}$, with $\omega_{\gamma\alpha\beta}=\lambda_\gamma^\mu \omega_{\mu\alpha\beta}=\Gamma_{\alpha\beta\gamma}$, satisfy $\nabla^\mu \omega_{\mu\alpha\beta}=0$.

$$D_0 a_b-D_c K_{cb}=\left(a_c-2n_c\right)K_{cb}-\left(TrK\right)a_b \ ,$$

$$D_0 \omega_b+D_c N_{cb}=-\left(a_c-2n_c\right)N_{cb}-(TrK)\omega_b \ .$$

Nester gauge (Estabrook, et al), based on propagating tetrad frames with a "teleparallel" (zero curvature, non-zero torsion) connection,

$$D_0 a_b-D_c K_{cb}=-\left(\omega_c-2\Omega_c\right)N_{cb}+(TrN)\omega_b \ ,$$

$$D_0 \omega_b+D_c N_{cb}=-\left(\omega_c-2\Omega_c\right)K_{cb}-(TrN)a_b \ .$$

Equations for the tetrad vectors, derived from the commutators of the tetrad vectors expressed in terms of the connection coefficients.

Evolution:

$$D_0 A_a = a_a - \partial_a(\ln\alpha) - \left(K_{ac} + \varepsilon_{abc}\omega_b\right)A_c \ ,$$

$$D_0 e_a^k + \frac{\partial\beta^k}{\partial x^m} e_a^m = -\left(K_{ac} + \varepsilon_{abc}\omega_b\right)e_c^k \ .$$

Constraints:

$$\varepsilon_{cab} e_a^m \frac{\partial e_b^k}{\partial x^m} = N_{dc} e_d^k - (TrN)e_c^k + \varepsilon_{cab} A_a \left(K_{bd} + \varepsilon_{bdf}\omega_f\right)e_d^k \ ,$$

$$\varepsilon_{cab} e_a^m \frac{\partial A_a}{\partial x^m} = 2\Omega_c + A_d N_{dc} - (TrN)A_c + \varepsilon_{cab} A_a \left(\left(K_{bd} + \varepsilon_{bfd}\omega_f\right)A_d - a_b\right) .$$

Initial Value Problem

Assume tetrad worldlines are orthogonal to the t=0 hypersurface, $\boldsymbol{\lambda}_0 = \mathbf{n}$, so $A_a = 0$, $\boldsymbol{\lambda}_a = \mathbf{e}_a$.

1. Start with a spatial metric h_{ij} and an extrinsic curvature K_{ij}, obtained by standard methods in the coordinate framework. Construct orthonormal triad fields e_a^k in the initial hypersurface. Then $K_{ab} = e_a^m e_b^n K_{mn}$, and the N_{ab} are determined from the commutators of the e_a^k.
2. The 3D Nester teleparallel gauge conditions, $\mathrm{Tr}N = 0$, $\nabla\times\mathbf{n} = 0$, can be used to fix the initial spatial orthonormal triad. Under a conformal transformation of the coordinate metric, $h_{ij} = \phi^4 \tilde{h}_{ij}$, $e_a^k = \phi^{-2}\tilde{e}_a^k$, $n_a = \phi^{-2}\tilde{n}_a - D_a\left(\ln\phi\right)$, the symmetric part of N_{ab} scales as ϕ^{-2}, so the Nester conditions are conformally invariant. A "bare" 3-geometry has $n_a = 0$ and $\mathrm{Tr}N = 0$.

Pseudo-hyperbolic and hyperbolic systems

If one pretends that the D_a are purely spatial directional derivatives, the evolution equations in the fixed, Lorentz, and Nester gauges have a very simple symmetric hyperbolic structure, with all propagation at light speed and variables coupled in pairs to form eigenvectors along each tetrad direction. However, the A_a will not stay zero, so the true hyperbolic structure includes the time derivatives hidden in the D_a.

A system of equations of the form

$$D_0\mathbf{q}+\mathbf{C}^f D_f\mathbf{q}=\mathbf{S}(\mathbf{q}) ,$$

where the $\mathbf{C}^f$ are characteristic matrices, is really

$$(\mathbf{I}+\mathbf{C}^f A_f)D_0\mathbf{q}+\mathbf{C}^f\partial_f\mathbf{q}=\mathbf{T}D_0\mathbf{q}+\mathbf{C}^f\partial_f\mathbf{q}=\mathbf{S}(\mathbf{q}) .$$

The true characteristic matrices are $\tilde{\mathbf{C}}^f=\mathbf{T}^{-1}\mathbf{C}^f$. Also, the nominal constraint equations are not the true constraint equations.

Fortunately, the $\mathbf{T}$ matrix has a simple structure, and its inverse can be found explicitly. It is block diagonal in groups of 8 variables labeled by the index c, $(N_{1c},N_{2c},N_{3c},a_c,K_{1c},K_{2c},K_{3c},\omega_c)$. The true hyperbolic system is still symmetric hyperbolic, with $\mathbf{T}$ as the symmetrizing matrix, as long as $A_aA_a<1$.

Care must be taken in the choice of the lapse in order that A_aA_a does not get too close to 1.

Plane waves, propagation in 1-direction:

Variables reduce to $N_{23},N_{32},N_{22}=-N_{33},N_{11},a_1,K_{22},K_{33},K_{23}=K_{32},K_{11},\omega_1,$ $A_1,e_2^2,e_2^3,e_3^2,e_3^3,e_1^1$. Physical modes $+(N_{23}+N_{32},K_{22}-K_{33}),\times(N_{22},K_{23})$; Constraint modes $(N_{23}-N_{32},K_{22}+K_{33})$; Longitudinal modes (a_1,K_{11}) and (N_{11},ω_1). Colliding circularly polarized waves can generate non-zero N_{11} and ω_1.

Spherical Symmetry

Need a *Cartesian* set of tetrad vectors to avoid singular twisting at the polar axis.

$$\mathbf{e}_1 = \sin\theta\cos\varphi e^{-\lambda}\frac{\partial}{\partial r} + \frac{\cos\theta\cos\varphi}{R}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{R\sin\theta}\frac{\partial}{\partial\varphi} ,$$

$$\mathbf{e}_2 = \sin\theta\sin\varphi e^{-\lambda}\frac{\partial}{\partial r} + \frac{\cos\theta\sin\varphi}{R}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{R\sin\theta}\frac{\partial}{\partial\varphi} ,$$

$$\mathbf{e}_3 = \cos\theta e^{-\lambda}\frac{\partial}{\partial r} - \frac{\sin\theta}{R}\frac{\partial}{\partial\theta} .$$

Only antisymmetric part of N_{ab} is non-zero. Connection variables are a_r, K_R, n_r, K_T. The metric variables are A_r, $e_r^r = e^{-\lambda}$, $e_\theta^\theta = \frac{1}{R}$. Note that $D_0 R = RK_T$, and $\partial_r R = 1 - Rn_r$. The Nester gauge evolution equation is $D_0 a_r = D_r K_R + \frac{2}{R}(K_R - K_T)$.

Hypersurface orthogonal gauge with dynamic lapse

Force congruence to be orthogonal to constant-t hypersurfaces. This implies $A_a \equiv 0$, $D_a = \partial_a$, $K_{ab} = K_{ba}$, and $a_b = D_b(\ln\alpha)$. I adopt a Bona-Masso type of dynamic lapse,

$$D_0(\ln\alpha) = f(\alpha)(TrK - K_0) .$$

This implies an evolution equation for a_a,

$$D_0 a_a = f\left[D_b K_{ab} - K_{ab}(2n_b) + \varepsilon_{abc}(K_{bd}N_{dc} - N_{db}K_{cd})\right] + \left(f + \alpha\frac{df}{d\alpha}\right)(TrK - K_0)a_a - (K_{ac} + \varepsilon_{abc}\omega_b)a_c ,$$

in which the vacuum momentum constraint has been used to give a form consistent with a symmetric hyperbolic system. The gauge condition on ω_a can be simply $\omega_a = 0$, or ω_a can be evolved using the Nester gauge evolution equation, say.

Hypersurface orthogonal hyperbolic system

The symmetry of K_{ab} can be enforced, reducing the number of variables by 3. Replace K_{ab} and K_{ba} by $\left(K_{ab}+K_{ba}\right)/\sqrt{2}$. If a_b is rescaled to $\tilde{a}_b \equiv a_b/\sqrt{f}$, the characteristic matrix is explicitly symmetric. The wave speeds are 0, $\pm\sqrt{f}$, $\pm\sqrt{\frac{1+f}{2}}$, and ± 1.
In spherical symmetry the wave speeds are $\pm\sqrt{f}$ and ± 1.

Einstein-Bianchi system

The several Einstein-Bianchi systems in the literature general involve large numbers of redundant variables, since not all symmetries of the Riemann tensor are enforced. The motivation for this is to ensure that all wave propagate at light speed.
However, there is no physical reason why waves other than the physical transverse-traceless modes should propagate at light speed. I argue that the number of extra variables and constraints should be minimized, since unenforced constraints mean more constraint-violating modes with the potential of being unstable. Starting from Choquet-Bruhat and York, I enforce all Riemann tensor symmetries and the Einstein equations to reduce the number of Riemann tensor variables to 10, corresponding to the number of degrees of freedom in the 5 complex scalars in the null tetrad decomposition of the Weyl tensor. The variables are the symmetrized electric and magnetic parts of the of the Riemann tensor, $E_{ab}=R_{0a0b}$ and $B_{ab}=\varepsilon_{acd}R_{cd0b}$.

The Bianchi identities evolve the Riemann tensor. The antisymmetric part of E_{ab} is zero and the antisymmetic part of B_{ab} is determined from the momentum constraint equations. The trace of E_{ab} is given by the energy constraint, and the trace of B_{ab} is zero, leaving 5 degrees of freedom in each.
Only the four physical transverse-traceless modes have wave speeds of ± 1. There are four mixed transverse-longitudinal modes with speeds $\pm 1/2$, and two longitudinal modes with speed zero.
The evolution equations for the connection coefficents are just the standard expressions for the E_{ab} and B_{ab} Riemann tensor components.
In the hypersurface-orthogonal gauge K_{ab} is symmetric,

$$D_0 K_{ab} = \frac{1}{2}\left(D_a a_b + D_b a_a - R_{0a0b} - R_{0b0a}\right) + \ldots,$$

and $D_0 N_{ab} + D_a \omega_b + \ldots = B_{ba}$.

Spherical Einstein-Bianchi

There is only one independent Riemann tensor component, R_{oror}, say, which propagates at zero speed, $D_0 E_R = -3K_T E_R$, $E_R = -2M/R^3$.
The connection coefficients a_r and K_R propagate at speed $\pm\sqrt{f}$, and n_r and K_T propagate at zero speed. This seems to make the evolution more stable, and constraint errors level out for stationary initial conditions, $f(\alpha) = \frac{2.8125}{\alpha\left(1+\alpha\right)}$, $K_0 = 0.48$, RK_T=-0.25 on the horizon, and the radial coordinate scaled so $e_r^r = \frac{1}{\sqrt{\alpha}}$ initially.
An important advantage of an approximately hyperbolic hypersurface and mostly zero-speed wave propagation is that almost all modes, and all the constraint-violating modes are outgoing at both boundaries.

Conclusions

- The tetrad formalisms have a formal relative simplicity, particularly in the nonlinear source terms. Almost everything can be derived by hand with a reasonable effort.
- The threading gauges are problematic, particularly in attempts to attain long time evolution in black hole contexts.
- The simplified Bianchi system with a hypersurface-orthogonal gauge and a dynamic lapse shows promise, but needs to be tested in 3D calculations.
- Much more work needs to be done on exploring various dynamic lapse conditions, and particularly dynamic shift conditions, to find conditions compatible with the tetrad framework which keep coordinates well behaved over long times in the vicinity of black hole horizons.