# Geometry of non-Markovian Gaussian processes

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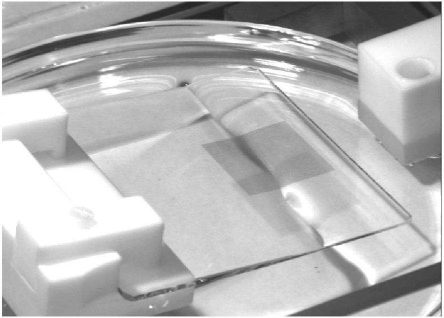
# References

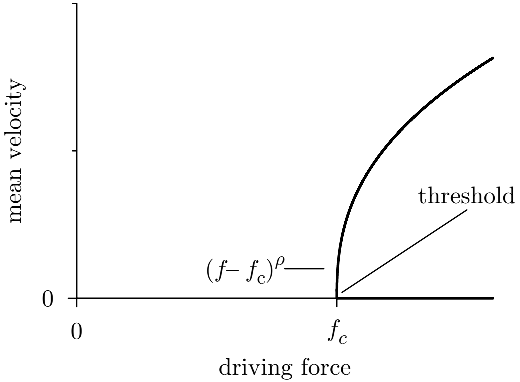

- A. Rosso, R. Santachiara, W. Krauth "Geometry of Gaussian signals" *J. Stat. Mech. Theory Exp.* L08001 (2005) and cond/mat-0609xxx
- S. Moulinet, A. Rosso, W. Krauth, E. Rolley "Width distribution of contact lines on a disordered substrate" Physical Review E **69** 035103 (2004)
- A. Rosso, W. Krauth, P. Le Doussal, J. Vannimenus, K. J. Wiese "Universal interface width distributions at the depinning threshold" Physical Review E **68** 036128 (2003)

# Experiment (Example)



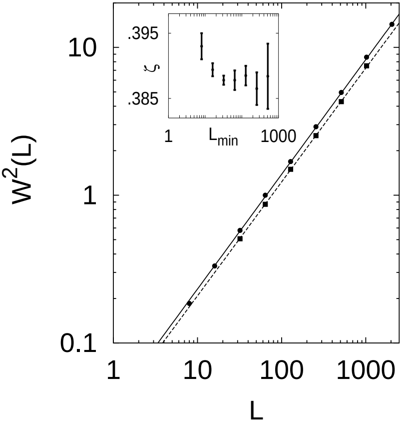
- Water or water/glycerol on “design sandpaper” on glass
- Interface close to depinning ($v \gtrsim 0$).
- Clean experiment, but restricted to small length scales (capillary length $\simeq$ 2mm)

# Order parameter at the depinning transition ($T = 0$)



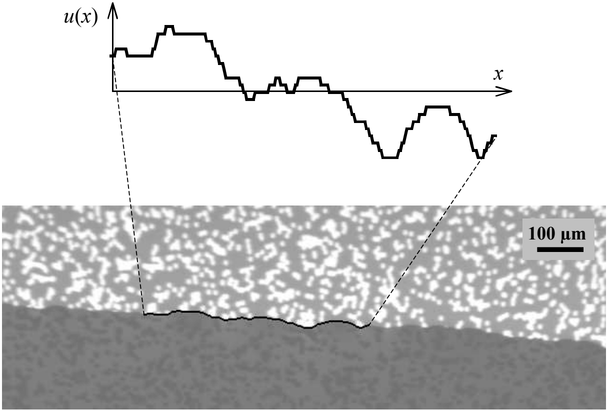

- mean velocity $v$ is order parameter of transition
- Control parameter: $v$ or $(f - f_c)$

# Roughness exponent; long-range (at threshold)




- Roughness exponent $\zeta$: (width) $\propto$ (length)$^{\zeta}$.
- $\zeta = 0.388 \pm 0.002$, governs wetting, and fracture

# Random geometry




- Roughness $\Longrightarrow$ geometry.

# Fourier-transformed line I

- Fourier-transform function $x(t)$ (always possible)

$$x(t) = \sum_{n=1}^{\infty} a_n \cos \frac{2\pi n}{L} t + b_n \sin \frac{2\pi n}{L} t$$

- Descriptions by $\{x(t)\}$ and by $\{a_n, b_n\}$ are equivalent
- Many experimental samples $\Longrightarrow$ probability distribution $P$

$$\left\{ \begin{array}{c} \text{fracture,} \\ \text{wetting,} \\ \text{domain wall} \end{array} \right\} \equiv P(a_1, a_2, \ldots; b_1, b_2, \ldots)$$

- Moments (equivalent information) $\langle \quad \rangle \equiv \langle \quad \rangle_{\text{samples}}$
  - $\langle a_1 \rangle, \langle a_2 \rangle, \ldots,$
  - $\langle a_1 a_2 \rangle, \langle a_2 a_3 \rangle, \ldots,$
  - $\langle a_1 a_2 a_3 \rangle, \langle a_2 a_3 a_4 \rangle, \ldots,$
  - $\langle a_1 a_2 a_3 a_4 \rangle, \langle a_2 a_3 a_4 a_5 \rangle, \ldots,$

# Fourier-transformed line II

- $x(t)$: Contact line, etc., at depinning for fixed disorder.
- Description of $x(t)$ using Fourier coefficients:

$$x(t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{L} t\right) + b_n \sin\left(\frac{2\pi n}{L} t\right)$$

- Simplest Ansatz:
  - ~~$\langle a_1 \rangle$~~, ~~$\langle a_2 \rangle$~~,..., 
  - $\langle a_1 a_1 \rangle, \langle a_1 a_2 \rangle, \ldots,$ nonzero
  - ~~$\langle a_1 a_2 a_3 \rangle$~~,~~$\langle a_2 a_3 a_4 \rangle$~~,...,
  - $\langle a_1 a_1 a_2 a_2 \rangle = \langle a_1 a_1 \rangle \langle a_2 a_2 \rangle$, etc..
- $\{a_k, b_k\}$: Gaussians (can choose them uncorrelated)
- contact-line physics: $\rightsquigarrow \langle a_1^2 \rangle, \langle a_2^2 \rangle, \ldots, \langle b_1^2 \rangle, \langle b_2^2 \rangle \ldots$ (?)

# Fourier-transformed line and roughness exponent

- Description of $x(t)$ using Fourier coefficients:

$$x(t) = \sum_{n=1}^{\infty} a_n \cos \frac{2\pi n}{L} t + b_n \sin \frac{2\pi n}{L} t$$

- $\{a_1, a_2, \ldots, b_1, b_2, \ldots\}$: independent Gaussian random variables
- Variances . . .

$$\sigma_n = \frac{1}{(\pi n)^{1/2}} \left( \frac{L}{2\pi n} \right)^{\zeta}$$

. . . to get roughness right. NB: $\langle a_n^2 \rangle = \langle b_n^2 \rangle = \sigma_n^2$

- Minimal model

# Algorithm for paths

- Generate paths with trivial algorithm

**for** $n = 1, 2, \ldots$ **do**

$$\left\{ \begin{array}{l} \sigma_n \leftarrow (\pi n)^{-\frac{1}{2}} \left(\frac{L}{2\pi n}\right)^{\zeta} \\ a_n \leftarrow \mathtt{gauss}\,(\sigma_n) \\ b_n \leftarrow \mathtt{gauss}\,(\sigma_n) \end{array} \right.$$

**for** $t = 0, \Delta_t, \ldots, L$ **do**

$$\left\{ \; x(t) \leftarrow \sum_{n=1}^{\infty} \left[ a_n \cos\left(2n\pi \frac{t}{L}\right) + b_n \sin\left(2n\pi \frac{t}{L}\right)\right] \right.$$

**output** $\{x(0), \ldots, x(L)\}$

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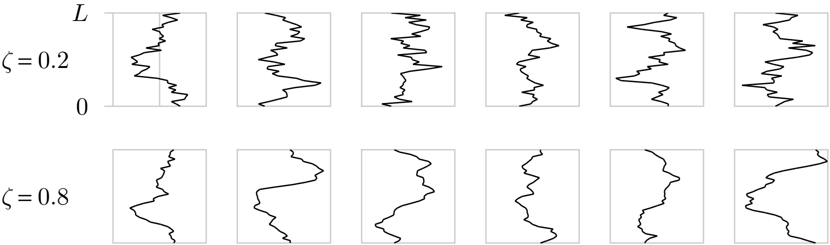
# Gaussian paths (periodic)

- Periodic Gaussian paths (40 modes, zero mean):




- Rough paths
  NB: rough (large $\zeta$) $\equiv$ smooth (on small scales)
- Anything to do with experimental paths ?
- Some paths are wider than other: width distribution

# Width distribution

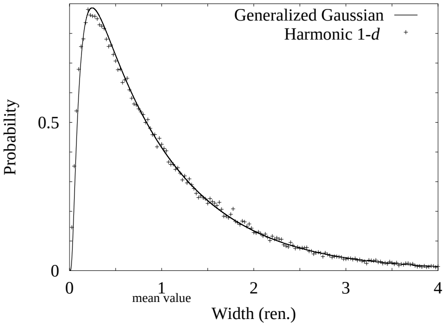
- Width of path

$$\omega_2 = \frac{1}{L}\int_0^L \mathrm{d}x \; x^2(t)$$

- Distribution $P(\omega_2)$, even in limit $L \to \infty$
- $P(\omega_2)$: analytical (or by simulation)

# Gaussian paths and simulation paths (periodic)

- Compare paths at depinning (one, two, three $d$, short range, long range) with Gaussian paths of same $\zeta$:




- Need $\gtrsim 1 \times 10^4$ samples for significant difference
- $\equiv$ Kurtosis very small
- FRG provides partial explanation

# Gaussian paths and contact lines (nonperiodic)

- Experimental paths are nonperiodic,

Traditionally, “free” paths are generated by a cosine series

$$x(t) = \sum_{n=1}^{\infty} c_n \cos\left(n\pi \frac{t}{L}\right).$$

- Zero mean, zero derivative at $t = 0$ and at $t = L$.

# Algorithm for “free” paths

- Generate also “free” paths with trivial algorithm:

**for** $n = 1, 2, \ldots$ **do**

$$\left\{ \begin{array}{l} \sigma_n \leftarrow \frac{2}{\pi n} \left(\frac{L}{\pi n}\right)^{\zeta} \\ c_n \leftarrow \texttt{gauss}(\sigma_n) \end{array} \right.$$

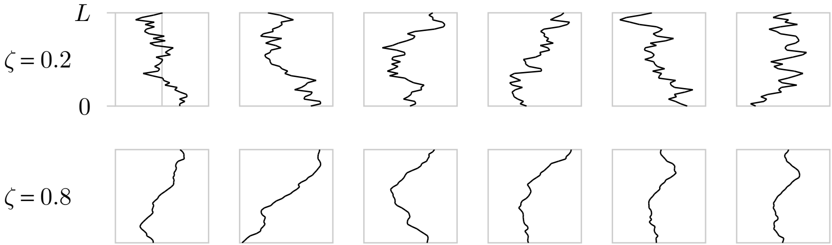
**for** $t = 0, \Delta_t, \ldots, L$ **do**

$$\{ \; x(t) \leftarrow \sum_{n=1}^{\infty} c_n \cos\left(n\pi \frac{t}{L}\right)$$

**output** $\{x(0), \ldots, x(L)\}$

# Gaussian paths ("free")

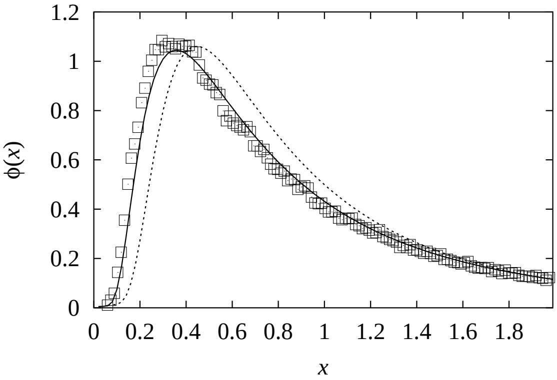
- Nonperiodic Gaussian paths (40 modes, zero mean):




- Anything to do with experimental paths ?.
- Width distribution differs from that of periodic paths.

# Gaussian paths and experimental paths (nonperiodic)

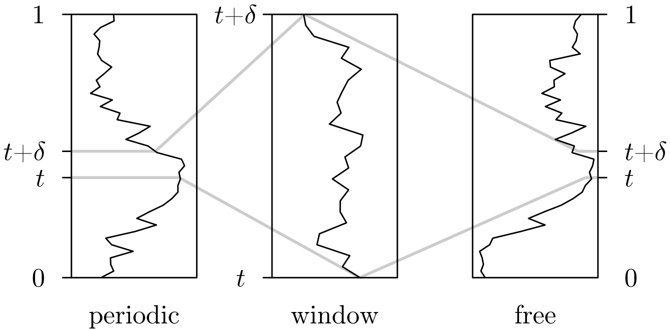
- Fit the distribution for contact lines with width distribution (fit parameter: $\zeta$).



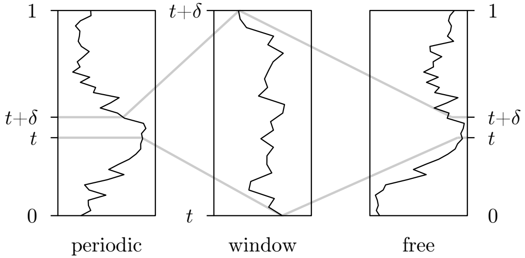

- $\zeta = 0.505$ much better than $\zeta = 0.388$.
- Roughness exponent without extrapolation of $\omega_2(L)$.
- Gaussian width distribution analytically (or from simulation)

# Periodic, “free”, window paths

- de Queiroz (2005): “window” $\neq$ “free”.

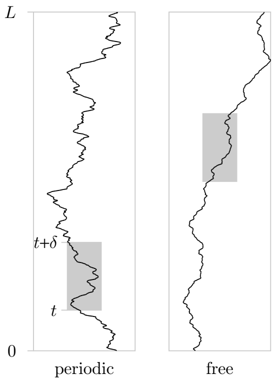


# Periodic, free, window paths




- zero-derivative condition at $t = 0$, $t = L$ influences statistics of free path
- exception: $\zeta = 0.5$ (random walk, Markov chain)
- intricate expansion in $\delta$ of width distribution (Rosso, Santachiara, W.K. 2005)…
- independent of boundary condition for $\delta \to 0$ ($\delta \ll t \ll L$)

# More windows




$$\langle x \rangle_{t,\delta} = \frac{1}{\delta} \int_t^{t+\delta} \mathrm{d}t \; x(t'),$$

$$\omega_2(t,\delta) = \int_t^{t+\delta} \mathrm{d}t' \; \left[ x(t') - \langle x \rangle_{t,\delta} \right]^2$$

# Elements of calculation

- for the “free” path (cosine series)

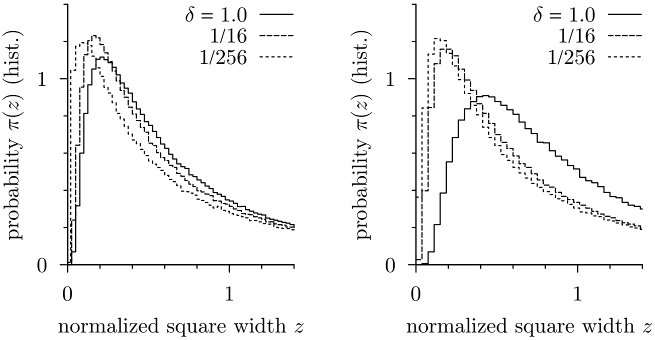
$$\omega_2(t,\delta) = \sum_{n,m=1}^{\infty} c_n c_m D_{nm}(t,\delta),$$

- where

$$D_{nm}(t,\delta) = \frac{1}{\delta}\int_t^{t+\delta} \mathrm{d}t' \; \overbrace{\cos\left(n\pi t'\right)\cos\left(m\pi t'\right)}^{\frac{1}{2}\{\cos[(m-n)\pi t']+\cos[(m+n)\pi t']\}} - \frac{1}{\delta^2}\left[\int_t^{t+\delta} \mathrm{d}t' \; \cos\left(n\pi t'\right)\right]\left[\int_t^{t+\delta} \mathrm{d}t' \; \cos\left(m\pi t'\right)\right].$$

- Width distribution:
  - Determine $D_{nm}(t,\delta)$ (once and for all)
  - Generate $\{c_1, c_2, \ldots\}$ (for each sample), get $\omega_2(t,\delta)$

# “Free”, “periodic” window width distribution




- Here, $\zeta = 0.75$.
- The two distributions agree for $\delta \to 0$ (rigorous)

# Independence of boundary condition

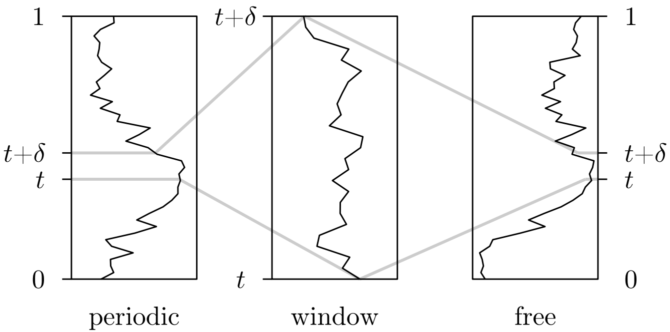
- $D_{nm}(x,\delta)$ function of $n\delta$, $m\delta$, can be handled by Euler–Maclaurin type formula:

$$\sum_{n=1}^{\infty} \frac{f(n\delta)}{n^\alpha} = \delta^{\alpha-1} \int_0^\infty \mathrm{d}t \left[ \sum_{m=\lfloor\alpha\rfloor}^{\infty} \frac{f^m(0) t^{m-\alpha}}{m!} \right], + \sum_{m=0}^{\infty} \delta^m f^m(0) \frac{\zeta(\alpha-m)}{m!}$$

- First term: naive integral limit
  Second term: naive Taylor expansion around zero.

# Non-Markovian character

- Random walk ($\zeta = 0.5, \alpha = 2$) is a misleading exception:




- Independent Fourier modes $\{c_1, c_2, \ldots\}$
- Also: independent increments $\Delta_x = x_k - x_{k-1}$
- Window = “free”
- Period $\neq$ “free”

# Summary

- Algorithms ...
- Geometry of paths $\simeq$ minimal Gaussian model (almost exact).
- Roughness exponents without extrapolation
- Width distributions:
  - full interval: free $\neq$ periodic; analytical (series)
  - free $\neq$ window; independence of boundary conditions for $\delta \to 0$ away from boundaries (analytic)
  - $P(\omega_2)$ has not been calculated analytically in $\delta \to 0$ limit (for $\zeta < 1$).
- No simple proof of $\delta \to 0$ universality.