Solvable Critical Dense Polymers

by

Jørgen Rasmussen

based on

- P.A. Pearce, J. Rasmussen, Solvable Critical Dense Polymers, hep-th/0610273
Outline

• Lattice model
• Planar Temperley-Lieb algebra
• Double-row transfer matrix and link states
• Inversion identity
• Finite-size corrections
• Physical combinatorics: selection rules and characters
• Relation to Dyck paths
• Hamiltonian limits
• Jordan cells and fusion
Lattice Model and Planar TL Algebra

Typical Configuration  –dense loop representation–

–with some boundary conditions

Face Operators

\[ u = \sin u + \cos u \]

- Spectral parameter: \( u \)
- Crossing parameter: \( \lambda = \frac{\pi}{2} \)
- Fugacity: \( \beta = 2 \cos \lambda = 0 \quad \rightarrow \quad \text{NO LOOPS} \)
**Local Properties**

**Inversion Relation**

\[
\begin{align*}
v & \rightarrow -v \\
\cos v \cos(-v) & \quad + \quad \beta \sin v \sin(-v) \\
\cos v \sin(-v) & \quad + \quad \sin v \cos(-v) \\
\cos^2 v & \end{align*}
\]

**Yang-Baxter Equation**

\[
\begin{align*}
u - v & \rightarrow v \\
u & \rightarrow u \\
\end{align*}
\]
Double-Row Transfer Matrix

Definition as $N$-tangle

\[ D(u) = \frac{1}{\sin 2u} \]

\[
\begin{array}{cccc}
\frac{\pi}{2} - u & \ldots & \ldots & \frac{\pi}{2} - u \\
u & \ldots & \ldots & u
\end{array}
\]

Normalization

\[ \lim_{u \to 0} D(u) = I = 
\]

Commuting Family

\[ D(u)D(v) = D(v)D(u) \]

- Multiplication is vertical concatenation of diagrams
- Equality is the equality of $N$-tangles

Crossing Symmetry

\[ D(u) = D(\lambda - u) \]
**Link States**

- In a ‘fixed direction’, the planar $N$-tangles can act on a vector space of link diagrams.
- Number of nodes: $N$
- For $N = 6$ and no defects, there is a basis of 5 link states:

```
1 2 3 4 5 6
```

**Defects**

- Number of defects: $\ell$
- For $N = 4$ and $\ell = 2$, there is a basis of 3 link states:

```
1 2 3 4
```

- By construction: $N - \ell \equiv 0 \pmod{2}$

**Example**

Initial state: ![Initial State Diagram]

Resulting state: ![Resulting State Diagram]

- Defects can be annihilated in pairs but not created under the action of TL.
Upper Block-Triangular Matrix Representation

Example ($N = 4$)

\[ D(u) = \begin{bmatrix}
\ast & \ast & \ast \\
0 & 0 & \ast \\
0 & 0 & \ast \\
\end{bmatrix} \]

- Individually, the blocks on the **diagonal** can be diagonalized.
- In general, $D(u)$ is non-diagonalizable.
Inversion Identity

\[ D(u)D(u + \frac{\pi}{2}) = \left(\frac{\cos^2 2N \times u - \sin^2 2N \times u}{\cos^2 u - \sin^2 u}\right)^2 I \]

Proof

\[ D(u)D(u + \frac{\pi}{2}) = -\frac{1}{\sin^2 2u} \]

where

\[ \frac{\pi}{2} - u \]

\[ \frac{\pi}{2} + u \]

\[ \frac{\pi}{2} - u \]

\[ \frac{\pi}{2} + u \]

\[ u \]

\[ \frac{\pi}{2} + u \]

\[ \frac{\pi}{2} - u \]

\[ u \]

\[ \frac{\pi}{2} + u \]

\[ \frac{\pi}{2} - u \]
\[
= \frac{1}{\cos^2 2u} \left\{ -\sin^2 2u + \sin 2u \cos 2u - \cos 2u \sin 2u + \cos^2 2u \right\}
\]

Certain half-arcs propagate since

\[
\frac{\pi}{2} + u = -\sin u \cos u + \cos u \sin u - \sin^2 u + 0 = -\sin^2 u
\]

and similarly

\[
\frac{\pi}{2} + u = \cos^2 u, \quad \frac{\pi}{2} - u = \cos^2 u, \quad \frac{\pi}{2} - u = -\sin^2 u
\]
We have

\[-u \quad \pi - u \quad \pi + u \quad u\]

\[= \sin^{4N} u + \cos^{4N} u\]
This implies

\[ u \pi^2 + u \pi^2 - u = (\cos^{4N} u + \sin^{4N} u) \]

\[ + (\cos u \sin u)^{2(N-1)} \]

where

\[ = -\cos^2 u \sin^2 u \quad \text{or} \quad = -\cos^2 u \sin^2 u \]
Eigenvalues

Functional Relation

\[ D(u)D(u + \frac{\pi}{2}) = \left( \frac{\cos^{2N} u - \sin^{2N} u}{\cos^{2} u - \sin^{2} u} \right)^{2} \]

subject to

\[ D(0) = 1, \quad D\left( \frac{\pi}{2} - u \right) = D(u) \]

General Solution

\[
D(u) = \begin{cases} 
\frac{N}{2^{N-1}} \prod_{j=1}^{N-1} \left( \frac{1}{\sin \frac{j\pi}{N}} + \epsilon_j \sin 2u \right) \left( \frac{1}{\sin \frac{2j\pi}{2N}} + \mu_j \sin 2u \right), & \text{N even} \\
\frac{1}{2^{N-1}} \prod_{j=1}^{N-1} \left( \frac{1}{\sin \frac{(2j-1)\pi}{2N}} + \epsilon_j \sin 2u \right) \left( \frac{1}{\sin \frac{2(2j-1)\pi}{2N}} + \mu_j \sin 2u \right), & \text{N odd}
\end{cases}
\]

where \( \epsilon_j^2 = \mu_j^2 = 1 \) for all \( j \).

- Number of link states with \( \ell \) defects: \( \left( \frac{N}{N-\ell} \right) - \left( \frac{N}{N-\ell-2} \right) \) → Selection Rules
- Ground state: \( \epsilon_j = \mu_j = 1 \) for all \( j \).
- Excitations: some \( \epsilon_j, \mu_j = -1 \).
Finite-Size Corrections

- The partition function for an $N' \times N$ strip reads

$$Z_{N,N'} = \text{Tr} \, D(u)^{N'} = \sum_n D_n(u)^{N'} = \sum_n e^{-N'E_n(u)}$$

where an eventual conformal invariance dictates that

$$E_n(u) = -\ln D_n(u) \simeq 2N f_{bulk} + f_{bdy} + \frac{2\pi \sin 2u}{N} \left(-\frac{c}{24} + \Delta + k\right)$$

- The bulk and boundary free energies are

$$f_{bulk} = \ln \sqrt{2} - \frac{1}{\pi} \int_0^{\pi/2} \ln \left(\frac{1}{\sin t} + \sin 2u\right) dt, \quad f_{bdy} = \ln(1 + \sin 2u)$$

- The conformal corrections also follow from the Euler-Maclaurin formula and one finds

$$c = -2, \quad \Delta = \Delta_{1,s} = \frac{(2 - s)^2 - 1}{8}, \quad s = 1, 2, 3, \ldots$$

where $s = \ell + 1$. This corresponds to the first column in the extended Kac table.

- The excitations can be organized in finitized characters

$$\chi_{1,s}^{(N)}(q) = q^{-\frac{c}{24} + \Delta_{1,s}} \left[\frac{N}{N-s+1}\right]_q - q^s \left[\frac{N}{N-s-1}\right]_q$$

where $q = \exp\left(-\frac{2\pi N' \sin 2u}{N}\right)$ while $\left[\frac{n}{m}\right]_q$ is a $q$-binomial.
Selection Rules

- The selection rules can be described using **physical combinatorics** related to the patterns of zeros of $D(u)$ in the complex $u$-plane.
- We recall that $D(u) \sim \prod_j \left( \frac{1}{\sin t_j} + \epsilon_j \sin 2u \right) \left( \frac{1}{\sin t_j} + \mu_j \sin 2u \right)$ where $t_j = j\pi/N$ for $N$ even, while $t_j = (2j - 1)\pi/2N$ for $N$ odd.
- With a period of $\pi$, the zeros are given by

  $$u \in \{(2 + \nu_j)\frac{\pi}{4} \pm y_j\}, \quad \nu_j \in \{\epsilon_j, \mu_j\}, \quad y_j = \frac{i}{2} \ln \tan \frac{t_j}{2}$$
Double-Column Configurations

- Height: $M = 6$
- Signature: $L = (5, 4, 1), R = (6, 4, 2, 1)$
- Occupancy: $m = |L| = 3, n = |R| = 4$
- Partial ordering: $S \preceq S' \text{ if } S_j \leq S'_j, j = 1, \ldots, |S|$

**Admissible**

(L, R) is admissible if $L \preceq R$ — in particular, $0 \leq m \leq n \leq M$

- Geometrically: signature links have non-negative slopes.
- Weight:

$$w(L, R) = \sum_{j=1}^{m} L_j + \sum_{j=1}^{n} R_j = 23$$

**Associated Monomial**

$$q^w = q^{23}$$
Generalized $q$-Narayana Numbers

Sum of Admissible Configurations

- For given $M$, $m$ and $n$, the sum of the monomials associated to the admissible double-column configurations is given by

$$
\langle M \rangle_{m,n} = q^{\frac{1}{2}m(m+1)+\frac{1}{2}n(n+1)} \left\{ \begin{array}{c} M \\ m, n \end{array} \right\}_q
$$

$$
= q^{\frac{1}{2}m(m+1)+\frac{1}{2}n(n+1)} \left[ \begin{array}{c} M \\ m \end{array} \right]_q \left[ \begin{array}{c} M \\ n \end{array} \right]_q - q^{n-m+1} \left[ \begin{array}{c} M \\ n+1 \end{array} \right]_q \left[ \begin{array}{c} M \\ m-1 \end{array} \right]_q
$$

- Fermionic – all coefficients are non-negative.

- For $0 \leq m \leq n \leq M$,

$$
\left\{ \begin{array}{c} M \\ m, n \end{array} \right\}_q = 1 + \mathcal{O}(q)
$$

$q$-Narayana Numbers

$$
N_q(M, m) = q^{m(m+1)} \left[ \begin{array}{c} M \\ m \end{array} \right]_q \left[ \begin{array}{c} M \\ m + 1 \end{array} \right]_q \frac{1 - q}{1 - q^M} = \langle M - 1 \rangle_{m, m}
$$
Characters

Finitized Characters from Selection Rules

\[
\chi_{1,s}^{(N)}(q) = \begin{cases} 
q^{1/2} \left( \sum_{m=0}^{N-s+1} \frac{N-2}{2} \frac{m+m+s-3}{2}/q + \sum_{m=0}^{N-s-1} \frac{N-2}{2} \frac{m+m+s-1}{2}/q \right), & s \text{ odd (} N \text{ even)} \\
q^{-1/2 - s/4 \frac{N-s+1}{2}} \sum_{m=0}^{N-1} \frac{N-1}{2} \frac{m+m+s-2}{2}/q^{-m}, & s \text{ even (} N \text{ odd)}
\end{cases}
\]

\[= q^{\Delta_{1,s} - \frac{c}{24}} \left( \left[ \frac{N}{N-s+1} \right]_q - q^s \left[ \frac{N}{N-s-1} \right]_q \right)\]

- In the continuum scaling limit, these finitized characters become the Virasoro characters

\[
\chi_{1,s}(q) = \lim_{N \to \infty} \chi_{1,s}^{(N)}(q) = q^{1/2} \frac{q^{\Delta_{1,s} - \Delta_{1,-s}} \prod_{n=1}^\infty (1 - q^n)}{q^{1/2} \prod_{n=1}^\infty (1 - q^n)} = q^{1/2} \frac{q^{\Delta_{1,s} (1 - q^s)} \prod_{n=1}^\infty (1 - q^n)}{q^{1/2} \prod_{n=1}^\infty (1 - q^n)}
\]

Generalization

\[
\chi_{r,s}(q) = \lim_{N \to \infty} \chi_{r,s}^{(N)}(q) = q^{1/2} \frac{q^{\Delta_{r,s} - \Delta_{r,-s}} \prod_{n=1}^\infty (1 - q^n)}{q^{1/2} \prod_{n=1}^\infty (1 - q^n)} = q^{1/2} \frac{q^{\Delta_{r,s} (1 - q^s)} \prod_{n=1}^\infty (1 - q^n)}{q^{1/2} \prod_{n=1}^\infty (1 - q^n)}
\]

\[
= q^{\Delta_{r,s} - \frac{c}{24}} \left( \left[ \frac{N}{N-s+r} \right]_q - q^r \left[ \frac{N}{N-s-r} \right]_q \right)
\]

\[
\rightarrow \left\langle \frac{M}{m,n;r}, \chi_{r,s}^{(N)}(q) = \text{[Fermionic expression as above]} \right\rangle
\]

\[
= q^{\Delta_{r,s} - \frac{c}{24}} \left( \left[ \frac{N}{N-s+r} \right]_q - q^r \left[ \frac{N}{N-s-r} \right]_q \right)
\]
Relation to Dyck Paths

Example \((s = 6, \; N = 23)\)

\[
M = \frac{N + s - 1}{2}, \quad n = m
\]

\[
M = \frac{N - 1}{2}, \; \frac{N - 2}{2}, \quad n = m + \frac{s - 2}{2}, \; m + \frac{s - 3}{2}, \; m + \frac{s - 1}{2}
\]

up \leftrightarrow left •
valley \leftrightarrow left ○
down \leftrightarrow right •
peak \leftrightarrow right ○

\[
s - 1 \begin{cases} 
\frac{N - 1}{2}, \; \frac{N - 2}{2} \\
\frac{s}{2}, \; \frac{s + 1}{2}
\end{cases}
\]

\[
\begin{pmatrix}
\bullet \\
\circ
\end{pmatrix} - \begin{pmatrix}
\circ \\
\bullet
\end{pmatrix} = \begin{pmatrix}
\circ \\
\bullet
\end{pmatrix}
\]

-bijection-

-extends to \((r, s)\)
Hamiltonian Limits

Expansion of $D(u)$

$$D(u) = I - 2uH + O(u^2)$$

It follows that

$$-H = \begin{array}{c}
\text{[Diagram of a complex structure]} \\
\end{array} + \begin{array}{c}
\text{[Another diagram]} \\
\end{array} + \ldots + \begin{array}{c}
\text{[Yet another diagram]} \\
\end{array}$$

- In terms of the generators of the linear TL algebra, this corresponds to

$$H = -\sum_{j=1}^{N-1} e_j$$

Defects Revisited

$$\begin{array}{c}
\text{[Diagram of two defect structures]} \\
\end{array} \sim \begin{array}{c}
\text{[Another diagram]} \\
\end{array}$$
Example: \((1, 2) \otimes f (1, 2)\)

\[
H \otimes f (1, 2) = H (1, 1) + H (1, 3)
\]

- For \(N = 4\), there are 5 link states:

Jordan Decomposition

\[
H_{(1,2)}(1,2) = - \begin{pmatrix}
0 & 2 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix} + \sqrt{2}I \rightarrow \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{2} & 0 & 0 \\
0 & 0 & 0 & \sqrt{3} & 1 \\
0 & 0 & 0 & 0 & \sqrt{3}
\end{pmatrix}
\]

- Jordan cells are present for the smallest system sizes which can accommodate them.
- If a Jordan cell corresponding to a particular energy level is present for a particular system size, a Jordan cell corresponding to the same energy level is present for all larger system sizes.
Finitized Dilatation Generator

\[ \mathcal{H}_{(1,2)} \rightarrow L^0_{(1,2)} = \text{diag} \left[ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 1, \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \right] \]

Finitized Partition Function

\[ Z^{(4)}_{(1,2)}(q) = \chi^{(4)}_{(1,1)}(q) + \chi^{(4)}_{(1,3)}(q) = q^{1/12} \left[ (1 + q^2) + (1 + q + q^2) \right] = q^{1/12} (2 + q + 2q^2) \]

Fusion Rules

\[ (1, 2) \otimes_f (1, 2) = (1, 1) \oplus_i (1, 3) = \mathcal{R}_{1,1} \]

and likewise

\[ (1, 2) \otimes_f (1, 3) = (1, 2) \oplus (1, 4) \]
\[ (1, 2) \otimes_f \mathcal{R}_{1,1} = (1, 2) \oplus (1, 2) \oplus (1, 4) \]
\[ \mathcal{R}_{1,1} \otimes_f \mathcal{R}_{1,1} = \mathcal{R}_{1,1} \oplus \mathcal{R}_{1,1} \oplus \mathcal{R}_{2,1} \]
\[ : \]

Fusion Algebra
Concluding Remarks

Summary

• Exactly solvable lattice model for critical dense polymers
• Finite-size corrections
• CFT with spectrum $c = -2$, $\Delta = \Delta_{1,s}$, $s = 1, 2, 3, \ldots$
• Physical combinatorics
• Link to Dyck paths
• Jordan cells
• Diagrammatic implementation of fusion

Outlook

• Other geometries: cylinder and torus
• Other models: critical percolation, logarithmic Ising model, . . .