Stanley, me and Life on the Light-Cone

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In 1981 Michael Green, John Schwarz and I computed the four-point one-loop S-matrix element for $N=4$ Yang-Mills and $N=8$ Supergravity and found that it is given by a box-diagram with kinematical factors.

It looked to me as if there was a scalar field theory behind.

How can one describe these theories with a scalar field? (Wrong but useful idea!)
N=4 in the light-cone gauge

Make the gauge choice $A^+ = 0$

Choose $x^+ = \frac{1}{\sqrt{2}} (x^0 + x^3)$ as the time.

We can then solve for $A^-$ since it satisfies a kinetic equation of motion and linearly combine the transverse physical degrees of freedom as

$$A = \frac{1}{\sqrt{2}} (A^1 + iA^2) \text{ and its c.c}$$

For the fermions we choose

$$\Psi = \frac{1}{2} \gamma_+ \gamma_- \Psi + \frac{1}{2} \gamma_- \gamma_+ \Psi = \Psi_+ + \Psi_-.$$
Similarly $\Psi_-$ satisfies a kinetic equation of motion and can be eliminated and the two-component $\Psi_+$ can be written as a complex Grassmann odd field $\Psi$.

We can now introduce a superspace

$$x^\pm, \quad x, \quad \bar{x}, \quad \theta^m, \quad \bar{\theta}_n$$

and span the N=4 supersymmetry

$$\{Q^m_+, \bar{Q}_{+n}\} = -\sqrt{2}\delta^m_n P^+$$
$$\{Q^m_-, \bar{Q}_{-n}\} = -\sqrt{2}\delta^m_n P^-$$
$$\{Q^m_+, \bar{Q}_{-n}\} = -\sqrt{2}\delta^m_n P,$$

When we act straight on a field we write $q$
The kinematical $q$'s will be represented by

\[
q^m_+ = -\partial^m + \frac{i}{\sqrt{2}} \theta^m \partial^+ , \quad \bar{q}^+_n = \bar{\partial}_n - \frac{i}{\sqrt{2}} \bar{\theta}_n \partial^+ ,
\]

and the dynamical ones as

\[
q^m_- = \bar{\partial}^m q^m_+ , \quad \bar{q}^-_m = \partial^m \bar{q}^+_m .
\]

On this space we can also represent "chiral" derivatives anticommuting with the supercharges $Q$.

\[
d^m = -\partial^m - \frac{i}{\sqrt{2}} \theta^m \partial^+ , \quad \bar{d}^n = \bar{\partial}^n + \frac{i}{\sqrt{2}} \bar{\theta}^n \partial^+ .
\]

To find an irreducible representation we have to impose the the chiral constraints
\[ d^m \phi = 0 ; \quad \bar{d}_m \bar{\phi} = 0 , \]

on a complex superfield \( \phi(x^\pm, x, \bar{x}, \theta^m, \bar{\theta}_n) \). The solution is then that

\[ \phi = \phi(x^+, y^- = x^- - \frac{i}{\sqrt{2}} \theta^m \bar{\theta}_m, x, \bar{x}, \theta^m). \]

It is particularly interesting to study the cases \( N = 4 \times \text{integer} \). For those values one can impose a further condition on the superfield \( \phi \) namely the "inside out" condition

\[ \bar{d}_{m_1} \bar{d}_{m_2} \ldots \bar{d}_{m_{N/2-1}} \bar{d}_{m_{N/2}} \phi = \]
\[ \frac{1}{2} \epsilon_{m_1 m_2} \ldots \epsilon_{m_{N-1} m_N} d^{m_{N/2+1}} d^{m_{N/2+2}} \ldots d^{m_{N-1}} d^{m_N} \bar{\phi} \]
\( N = 4 \)

\[
\phi(y) = \frac{1}{\partial^+} A(y) + \frac{i}{\sqrt{2}} \theta^m \theta^n \overline{C}_{mn}(y) \\
+ \frac{1}{12} \theta^m \theta^n \theta^p \theta^q \epsilon_{mnpq} \partial^+ \overline{A}(y) \\
+ \frac{i}{\partial^+} \theta^m \overline{\chi}_m(y) + \frac{\sqrt{2}}{6} \theta^m \theta^n \theta^p \epsilon_{mnpq} \chi^q(y).
\]

The full action could then be found as

\[
S = -\int d^4 x \int d^4 \theta d^4 \overline{\theta} \\
\left\{ \frac{\overline{\phi}^a}{\partial^+} \overline{\phi}^a + \frac{4g}{3} f^{abc} \left( \frac{1}{\partial^+} \overline{\phi}^a \phi^b \overline{\phi}^c + \text{c.c.} \right) \\
- g^2 f^{abc} f^{ade} \left( \frac{1}{\partial^+} (\phi^b \partial^+ \phi^c) \frac{1}{\partial^+} (\overline{\phi}^d \partial^+ \overline{\phi}^e) \\
+ \frac{1}{2} \phi^b \overline{\phi}^c \phi^d \overline{\phi}^e \right) \right\}.
\]

With this action we (Brink, Lindgren and Nilsson 1982) proved that the perturbation expansion is finite.
We also realized that the maximal supergravity could be written in this way

\( N = 8 \)

\[ \phi(y) = \frac{1}{\partial^2 + 2} h(y) + i \theta^m \frac{1}{\partial^2 + 2} \bar{\chi}_m(y) \]
\[ \ldots + \theta^{mnp} \bar{C}_{mnp}(y) \]
\[ \ldots + \bar{\theta}^{(7)} \partial^+ \chi^m(y) + \bar{\theta}^{(8)} \partial^2 \bar{h}(y), \]

How do we construct the interacting theory?

We will only consider massless theories so we solve the condition \( p^2 = 0 \). We then find

\[ p^- = \frac{p \bar{p}}{p^+}. \]

The generator \( p^- \) is really the Hamiltonian.
We have to find representations to the super-Poincaré algebra.

Generators that involve the "time" are called dynamical (or Hamiltonians) and the others kinematical.

The dynamical ones are non-linearly realized. We have to construct all of them.

The hard ones are rotations into "time". The linear part is

\[
j^- = i x \frac{\partial \bar{\theta}}{\partial^+} - i x^- \partial + i \left( \theta^\alpha \bar{\theta}_\alpha + \frac{i}{4\sqrt{2} \partial^+} (d^\alpha \bar{d}_\alpha - \bar{d}_\alpha d^\alpha) \right) \frac{\partial}{\partial^+}
\]
The \( N = 8 \) Supergravity action to first order is then

\[
\int d^4x \int d^8\theta d^8\bar{\theta} \mathcal{L} \equiv \int \mathcal{L},
\]

where,

\[
\mathcal{L} = -\bar{\phi} \frac{\Box}{\partial^4} \phi + \left( \frac{4\kappa}{3\partial^4} \bar{\phi} \bar{\partial} \partial \phi \partial^{+2} \phi - \frac{4\kappa}{3\partial^4} \bar{\phi} \bar{\partial} \partial^{+} \phi \bar{\partial} \partial^{+} \phi + \text{c.c.} \right)
\]

How do we construct the four-point function? We can do it by trial and error. Too hard.

Instead we found a remarkable property of maximally supersymmetric theories. (with Ananth and Ramond)
The Hamiltonian as a Quadratic Form

The usual relation is that

\[ H = \frac{1}{4} \{ Q^m, Q^-m \} \]

For both \( N = 4 \) and \( N = 8 \)

\[ H = \int \delta\bar{q}^-m \bar{\phi} \delta q^-m \phi \]

Not an anticommutator, but a quadratic form.

With this form we could run a Mathematica program comparing with the four-point function of gravity.

The result was a four-point coupling with 96 terms. (In the covariant form there are about 5000 terms.) with Ananth, Heise and Svendsen.
Higher Symmetries for $N = 4$ Yang-Mills Theory

We know that the $d = 4$ theory is conformally invariant, i.e. under $PSU(2,2|4)$ even for the quantum case. We can in fact construct the whole theory by closing the conformal algebra by guessing the correct dynamical supersymmetry generator $Q_-$. 
Higher Symmetries for $N = 8$ Supergravity Theory

$N = 8$ Supergravity, unlike $N = 4$ Yang-Mills, is not superconformal invariant; however, it does have the non-linear Cremmer-Julia $E_{7(7)}$ symmetry.

**How do we implement the $E_{7(7)}$ symmetry?**

Go back to covariant component form (Cremmer, Julia and Freedman, de Wit)

$$\mathcal{L} = \mathcal{L}_S + \mathcal{L}_V + \mathcal{L}_{\text{others}}$$

$\mathcal{L}_S$ is a Coleman-Wess-Zumino non-linear Lagrangian. The $E_{7(7)}$ is clear.
$\mathcal{L}_V$ can be written as

$$\mathcal{L}_V = -\frac{1}{8} F^{\mu \nu ij} G_{\mu \nu}^{ij},$$

The Lagrangian is quadratic in the field strengths. Introduce the self-dual complex field strengths

$$F^{\mu \nu ij} = \frac{1}{2} (F^{\mu \nu ij} + i\tilde{F}^{\mu \nu ij})$$

and

$$G^{\mu \nu ij} = \frac{1}{2} (G^{\mu \nu ij} + i\tilde{G}^{\mu \nu ij})$$

The equations of motion are given by

$$\partial_\mu G^{\mu \nu ij} = \partial_\mu (G^{\mu \nu ij} + \tilde{G}^{\mu \nu ij}) = 0,$$

while the Bianchi identities read

$$\partial_\mu \tilde{F}^{\mu \nu ij} = \partial_\mu (F^{\mu \nu ij} - \tilde{F}^{\mu \nu ij}) = 0.$$
Assemble in one column vector with 56 complex entries

\[ Z^{\mu\nu} = \begin{pmatrix} G^{\mu\nu ij} + F^{\mu\nu ij} \\ G^{\mu\nu ij} - F^{\mu\nu ij} \end{pmatrix} \equiv \begin{pmatrix} X^{\mu\nu \ ab} \\ Y^{\mu\nu \ ab} \end{pmatrix}, \]

where \( a, b \) are \( SU(8) \) indices, with upper(lower) antisymmetric indices for 28(28).

This is a 56 under \( E_{7(7)} \).

The 70 transformations are

\[ \delta X^{\mu\nu \ ab} = \Xi^{abcd} Y^{\mu\nu \ cd}, \]
\[ \delta Y^{\mu\nu \ ab} = \Xi_{abcd} X^{\mu\nu \ cd}, \]
We now specialize to the **light-cone gauge**. We choose \( A^+ = 0 \) and solve for \( A^- \). We then make non-linear **field redefinitions**, \( A^{ij} \rightarrow B^{ij} \) and \( C^{ijkl} \rightarrow D^{ijkl} \) to get rid of "time" derivatives in the interaction terms.

This will mix up the fields and the Hamiltonian is no longer quadratic in \( B^{ij} \).

We can now read off the \( E_7(7)/SU(8) \) transformations in the vector and scalar fields.

**However, the other fields now take part in the transformations!**
The $\frac{E_7(7)}{SU(8)}$ quotient symmetry must commute with the other symmetries in particular with the supersymmetry. $[\delta_{70}, \delta_S]\varphi = 0$.

(There is no $E_{7(7)}$ supergroup.)

By using that we get the transformations for all fields in the multiplet.

How can $\frac{E_7(7)}{SU(8)}$ commute when $SU(8)$ does not, and

$[\delta_{70}, \delta_{70}] = \delta_{SU(8)}$?

Consider the Jacobi identity

$([[[\delta_{70}, \delta_{70}], \delta_S] + [[[\delta_S, \delta_{70}], \delta_{70}] + [[[\delta_{70}, \delta_S], \delta_{70}]])\varphi = 0$

Since $[\delta_S, \delta_{70}]\delta_{70}\varphi \neq 0$, it works! $\delta_{70}\varphi$ non-linear! We only claim that $[\delta_S, \delta_{70}]\varphi = 0$. All fields including the graviton transform under $\frac{E_7(7)}{SU(8)}$ and into each other.
Some of the transformations

Vectors:

$$\delta \overline{B}_{ij} = - \kappa \Xi^{klm} \left( \frac{1}{4} \overline{D}_{ijkl} \overline{B}_{mn} + \frac{1}{4!} \frac{1}{\partial} \overline{D}_{klmn} \partial^+ \overline{B}_{ij} - \frac{1}{4!} \epsilon_{ijklmns} \frac{1}{\partial} B^{rs} \partial^+ h ight)$$

$$+ \frac{i}{3!} \overline{D}_{klmn} \partial^+ \overline{X}_{ij} - \frac{i}{3!} \epsilon_{ijklmns} \frac{1}{\partial} \chi^{rst} \overline{\psi}_n$$

$$+ \kappa \Xi_{ijkl} \frac{1}{\partial^+} \left( \frac{1}{4} D_{klmn} \partial^+ \overline{B}_{mn} - \frac{1}{\partial^+} B_{kl} \partial^+ h ight)$$

$$+ \frac{i}{4(3!)} \overline{X}_{mnp} \chi^{rst} \epsilon^{klmnpqrst} - 3 i \frac{1}{\partial^+} \chi^{kln} \partial^+ \overline{\psi}_n \right)$$ \hspace{1cm} (1)

Gravitini:

$$\delta \overline{\psi}_i = - \kappa \Xi^{mnpq} \left( \frac{1}{4!} \epsilon_{mnpqrs} \partial^+ \overline{D}_{rstu} \overline{\psi}_u + \frac{1}{4!} \frac{1}{\partial} \overline{D}_{mnpq} \partial^+ \overline{\psi}_i + \frac{1}{4!} \overline{D}_{mnpq} \overline{\psi}_i ight)$$

$$- \frac{1}{4!} \epsilon_{mnpqrs} \frac{1}{\partial} \chi^{rst} \partial^+ h + \frac{1}{4!} \overline{X}_{imn} \overline{B}_{pq} + \frac{1}{4!} \frac{1}{\partial} \overline{X}_{mnp} \partial^+ \overline{B}_{iq} \right)$$ \hspace{1cm} (2)

Gravition:

$$\delta h = - \kappa \Xi^{mnpq} \left( \frac{1}{4!} \frac{1}{\partial} \overline{D}_{mnpq} \partial^+ h + \frac{1}{8} \overline{B}_{mn} \overline{B}_{pq} + \frac{i}{\partial} \overline{X}_{mnp} \overline{\psi}_q \right).$$ \hspace{1cm} (3)
We then find that we can write the order $\kappa$ transformation as

$$
\delta \varphi = \frac{\kappa}{4!} \equiv_{\text{mnpq}} \frac{1}{\partial + 2} \left( \dd d_m \dd d_n \dd d_p \dd d_q \frac{1}{\partial + 3} \varphi \right) + \ldots .
$$

This expression is in fact unique! It can be rewritten in a very efficient form

$$
\frac{\kappa}{4!} \equiv_{\text{mnpq}} \left( \frac{\partial}{\partial \eta} \right)_{\text{mnpq}} \frac{1}{\partial + 2} \left( e^{\eta \dd d} \varphi e^{\eta \dd d} \right)_{\eta = 0},
$$

where $\dd d = \dd d^{\partial+}$.
The Hamiltonian

We write

$$\delta_{s}^{\text{dyn}} \varphi = \delta_{s}^{\text{dyn}}(0) \varphi + \delta_{s}^{\text{dyn}}(1) \varphi + \delta_{s}^{\text{dyn}}(2) \varphi + \mathcal{O}(\kappa^{3})$$

We can now require

$$[\delta_{70}, \delta_{s}^{\text{dyn}}] \varphi = 0$$

Here we can use the inhomogeneity of the 70 transformation

$$[\delta_{70}(-1), \delta_{s}^{\text{dyn}}(2)] \varphi + [\delta_{70}(1), \delta_{s}^{\text{dyn}}(0)] \varphi = 0$$

This gives the order $\kappa^{2}$ dynamical supersymmetry. We can then use the quadratic form to find the Hamiltonian to order $\kappa^{2}$. Much simpler than before!
Possible counterterms for N=8

Let us check first in gravity. We can write the three point coupling as

\[ \delta_H^\kappa h = \kappa \partial^+ n \left[ e^{a \hat{\partial}^+ m} h e^{-a \hat{\partial}^+ m} h \right]_{a^2} \]

\[ = \kappa \partial^+ n \left( \frac{\partial}{\partial a} \right)^2 \left[ e^{a \hat{\partial}^+ m} h e^{-a \hat{\partial}^+ m} h \right]_{a^0} , \]

A possible one-loop counter term is

\[ \delta_H^{g_{1}} h = \kappa^3 \partial^+ n \left[ E \partial^+ m E^{-1} \partial^+ m h \right] _{a^3, b} , \]

\[ E = e^{a \hat{\partial} + b \hat{\partial}} \quad \text{and} \quad E^{-1} = e^{-a \hat{\partial} - b \hat{\partial}} , \]

Consistent with the algebra for two choices of m and n
This can in fact be generalized to all orders.

\[ \delta_{H}^{g_1} h = \kappa^{2l+1} \partial^+ \left[ E \partial^+ h E^{-1} \partial^+ h \right] \bigg|_{a^2+l, b^l}, \]

\[ \delta_{H}^{g_2} h = \kappa^5 \frac{1}{\partial^3} \left[ E \partial^{+4} h E^{-1} \partial^{+4} h \right] \bigg|_{b^6}. \]

There is another series starting with

\[ \delta_{H}^{g_2} h = \kappa^5 \frac{1}{\partial^3} \left[ E \partial^{+4} h E^{-1} \partial^{+4} h \right] \bigg|_{b^6}. \]

We are interested in counterterms which are non-zero when we use the equation of motion.

\[ \partial^- h = \delta_{H} h = \frac{\partial \bar{\partial}}{\partial^+} h + O(\kappa). \]

All but the third terms can be written as \( \Box (..h...h) \)
There are no three-point counter terms for $N = 8$

$$\delta_H \phi = \ldots (\ldots \phi \bar{\phi})$$

since the r.h.s. is not chiral.

When we consider the four-point coupling we have to use the $E_7(7)$ symmetry. Remember how we obtain the four-point coupling.

$$[\delta_{70}(-1), \delta_{s}^{\text{dyn}}(2)] \phi + [\delta_{70}(1), \delta_{s}^{\text{dyn}}(0)] \phi = 0$$

The terms talk to each other pairwise. They have the same number of derivatives.
A four-point counterterm $\delta_{s,c}^{dyn}(2)$ must satisfy

$$[\delta_{70}(-1), \delta_{s,c}^{dyn}(2)] \varphi = 0$$

Furthermore it has to satisfy all the commutations rules with the full $N = 8$ superalgebra. Well-defined problem but algebraically difficult. We still do not have the final result.

I wish you had been part of the collaboration, Stanley!

Congratulations to the first 80!