Yesterday: considered the geometry of $T \otimes T^\ast$.

We defined the notion of a generalized complex manifold: manifold $M$ of dimension $n$ together with $E \subset (T \otimes T^\ast) \otimes C$ so that

- $E \subset (T \otimes T^\ast) \otimes C$ is maximally isotropic
- $E \cap \bar{E} = 0$
- sections of $E$ are closed under the Courant bracket.

Basic examples:

- symplectic manifold: $E$ is the transform of $T \otimes T^\ast$ by $i\omega$
- complex manifold
  
  $E = T^{1,0} \otimes T^{0,1}$

- transform either of the above by a closed $B$-field

We would like to have an easy way to check the closedness of $E$ w.r.t. Courant bracket. This is possible if we use some facts about pure spinors.

**Pure spinors**: We interpreted $\Lambda T^\ast$ as

Spinors:

\[(X, z) \cdot \psi = i X \psi + \zeta \cdot \psi\]

Clifford multiplication
Suppose
\[(X + Z) \cdot η = 0 \implies (X + Z)^2 \cdot η = 0\]
\[\implies (X + Z, X + Z) = 0\]

Thus the annihilator of any spinor η ∈ Λ² T* is an isotropic subspace in
\[T ⊗ T^* \text{ (or } (T ⊗ T^*) ⊗ C)\].

**Def:** A spinor η is called a **pure spinor** if its annihilator is maximally isotropic.

**Note:** The property of being pure is invariant under the action of the spin group.

**Examples:**
- \[0 ∈ Λ⁰ T^* ⊂ Λ² T^*\]
  \[\implies i_x 1 + ζ ∧ 1 = 0 \iff ζ = 0\]
  hence annihilator \((1) = T ⊂ T ⊗ T^*\)
  i.e., 1 is a pure spinor
- \[ψ B, 1 \text{ is pure}\]
- If \(\dim M = n\), then decomposable \(m\)-forms are pure spinors:
  \[\bar{X} (dx_1 ∧ ... ∧ dx_m) + ζ ∧ (dx_1 ∧ ... ∧ dx_m) = 0\]
  \[X = \sum_{i=1}^n a_i \frac{∂}{∂x_i}\]
  \[ζ = \sum_{i=1}^n b_i dx_i\]
  \[ψ B, (dx_1 ∧ ... ∧ dx_m) \text{ is pure}\]
Proposition: There is a 1-to-1 correspondence between pure spinors (taken up to scalar multiplication) and maximally isotropic subspaces
\[ \mathcal{E} \rightarrow \mathcal{E}_{\mathcal{E}} \subset T \otimes T^* \]

Proof: F. Cartan, C. Chevalley.

Fact: \( \dim (\mathcal{E}_\mathcal{E} \cap \mathcal{E}_{\bar{\mathcal{E}}}) > 0 \) if and only if \( \langle \mathcal{E}, \bar{\mathcal{E}} \rangle = 0 \)
Here \( \langle , \rangle \) is the 'Mukai' pairing on spinors.

If now \( M \) is a generalized complex manifold, then \( \mathcal{E} \subset (T \otimes T^*) \otimes \mathcal{C} \) corresponds to some pure spinor \( \mathcal{E} \) and the fact above implies
\[ E \otimes \bar{E} = 0 \iff \langle \mathcal{E}, \bar{\mathcal{E}} \rangle \neq 0 \]

Note that if \( M \) is a generalized complex manifold, then for \( \mathcal{E} \in \mathcal{M} \) a 4-dim space of pure spinors \( \mathcal{E} \in \Lambda^* T^*_m \) s.t.
\[ \text{Ann} (\mathcal{E}_m) = \mathcal{E}_m. \]

This space \( \mathcal{E}_m \) fit together into a line bundle \( L \rightarrow M \) - the canonical bundle of the generalized complex structure.
Example: \( E = T^{0,1} \oplus T^{1,0} \ast \) 
(gives a usual complex structure) 
then \( L = K_M \).

Proposition: If \( \omega \) is a pure spinor, then 
\( E_\omega \) is closed under the Courant bracket. 
If \( d\omega = (x+\bar{z})\omega \) for some \( x, \bar{z} \). 
In particular, if \( \omega \) is a pure spinor and \( d\omega = 0 \), then \( E_\omega \) is closed under the Courant bracket.

Definition: A \underline{generalized Calabi-Yau manifold} 
is a manifold \( M^{2n} \) together with 
a closed form \( \omega \in \Lambda^{\ast} T^\ast M \) s.t. 
\( \ast \omega \) is a pure form 
\( \langle \omega, \bar{\omega} \rangle = 0 \).

Examples: (1) If \( M \) is an ordinary CY manifold 
(there this means a complex manifold 
\( M \) together with a section \( \omega \in H^0(K_M) \) 
trivializing \( K_M \)), 
\( \omega \) - top degree holomorphic form \( \Rightarrow \) decomposable 
\( \ast \omega \) - pure spinor 
\( \langle \omega, \bar{\omega} \rangle = i \Re \langle \omega, \bar{\omega} \rangle + 0 \).
2. If \((M, \omega)\) is a symplectic manifold, then
\[
\mathbb{C} = \mathbb{C}^\omega \quad \text{is a pure sym.}
\]
and
\[
\langle \mathbb{C}^\omega, \mathbb{C}^\omega \rangle = \langle \mathbb{C}^{2\omega}, \mathbb{I} \rangle = i (2\omega)^n \neq 0
\]

3. A B-field transform of a symplectic manifold is also a generalized \(\mathbb{C}^\varepsilon\):
\[
\mathbb{C}^{B+i\omega} = \mathbb{C}^B \quad \text{pure sym.}
\]

\[\underline{\text{Comment:}}\quad \text{At a first glance it sounds strange that one can treat symplectic and \(\mathbb{C}^\varepsilon\) manifolds on an equal footing.} \]

Indeed:

- If \((M, \omega)\) compact symplectic
  \(\Rightarrow\) Lie algebra automorphism group \(= C^\infty(M)/\mathbb{R}\)
  is infinite dimensional

- If \((M, \mathbb{R})\) compact \(\mathbb{C}^\varepsilon\) with \(b_1 = 0\)
  \(\Rightarrow\) \(M\) has no vector fields (holomorphic)

However, if we add the action of B-fields we get similar pictures:
If \((M, w)\) symplectic and we consider the action of \(\text{Diff}(M) \times \mathbb{R}^2\) closed on the level of Lie algebras, we get

\[
(L_x + d\bar{z})\, e^{i\bar{w}} = 0 \implies L_x i\bar{w} + d\bar{z} = 0
\]

\[
\implies L_x w = 0
\]

\[
d\bar{z} = 0
\]

i.e. we get again \(C^\infty(M) / \mathbb{R}\).

If \((M, R)\) - compact complex CY (assume also Kahler), we get

\[
(L_x + d\bar{z})\, \mathcal{E} = 0 \implies L_x \mathcal{E} = 0 \quad d\bar{z} \wedge \mathcal{E} = 0
\]

\[
\implies x = 0
\]

\[
d\bar{z} \text{ is of type } (1,1)
\]

\[
\implies \text{ by Ca} \text{alemma}
\]

\[
d\bar{z} = \partial \bar{\partial} f
\]

Hence we get again \(C^\infty(M) / \mathbb{R}\).

**Caution:** This gives Lie algebras which are isomorphic as vector spaces but have different Lie brackets:

- Poison bracket in the symplectic case
- and trivial bracket in the CY case
Question (Greg Moore): Can one incorporate quantum corrections on the symplectic side in order to make the Lie brackets match?

If $M^{4e}$ is hyperkähler we have 3 symplectic forms

$\omega_1, \omega_2, \omega_3$

$\omega_1$ - Kähler

$\omega_2 + i \omega_3$ - holomorphic symplectic

Now B-field

$\omega_2 + i \omega_3$ - symplectic

$\omega$ - generalized complex structure

The spinor

$e = t^\frac{\omega_2 + i \omega_3}{t}$

defines a B-field transform of a symplectic manifold

Now $e \xrightarrow{t \to 0} (\omega_2 + i \omega_3)^k$

pure spinor defining an ordinary CI structure.
Thus: Generalized complex structures allow us in some cases to interpolate between complex and symplectic manifolds.

As we mentioned before: If $\psi = \psi_0 + \psi_1 \cdots$ is a pure spinor, then $\psi_0 \neq 0$ and

$$\psi = \psi_0 \xi B + i \omega$$

for some $\xi, B, \omega$.

The GCY condition is simply that

$$d\psi = 0 \implies \psi_0 = \text{const.}$$

However for a general generalized complex manifold this is not the case and we can see a type change for our pure spinor from point to point on the manifold.

Type change for pure spinors - even case

$$L = \Lambda^{0,1} \otimes C \rightarrow \Lambda^{0,1} \otimes C$$

$$\psi \rightarrow \psi_0$$
to defines a section of \( L^* \) which vanishes where the type changes.

In particular if \( c_1(L) \neq 0 \) \( \Rightarrow \) type change must occur.

**Examples:**

1. \( M = \mathbb{R}^4 = \mathbb{C}^2 \)

\[ \eta = z_1 + dz_1 \wedge dz_2 = z_1 \exp \left( \frac{dz_1}{z_1} \wedge dar{z}_2 \right) \]

When \( z_1 \neq 0 \) \( \Rightarrow \) \( \eta \) = B-field transform of a \textit{complex} manifold

When \( z_1 = 0 \) \( \Rightarrow \) \( \eta = dz_1 \wedge d\bar{z}_2 \) = complex manifold

Note that this is a \textit{generalized complex} manifold. Indeed

- \( \eta \) - pure \( \Rightarrow \) \( \eta \) closed under the constant bracket i.e. \( d(f \eta) = 0 \) for some function.
- \( \langle \eta, \bar{\eta} \rangle = \langle z_1 + dz_1 \wedge dz_2, \bar{z}_1 + d\bar{z}_1 \wedge d\bar{z}_2 \rangle \)
  \[ = dz_1 \wedge d\bar{z}_2 \wedge dz_1 \wedge d\bar{z}_2 \neq 0 \]
- \( d(z_1 \eta) = 0 \) \( \Rightarrow \) \( \eta \) - integrable

2. If \( M \) is a complex surface with \( \beta \) a holomorphic section of \( K^{-1} \Rightarrow \beta \)

is a holomorphic 2-form outside \( \{ z_1 \neq 0 \} \).
Topological conditions: Assume that we have a generalized almost complex manifold \( M \) together with

- \( E \subset (\mathcal{T} \otimes \mathcal{T}^*) \otimes \mathcal{Q} \)
- \( E = \text{isotropic} \)
- \( E \cap \overline{E} = 0 \).

In other words we have an almost complex structure

\[ J : \mathcal{T} \otimes \mathcal{T}^* \rightarrow \mathcal{T} \otimes \mathcal{T}^* \]

and \( J \) is compatible with the indefinite metric on \( \mathcal{T} \otimes \mathcal{T}^* \). In other words \( M \) has (generalized almost complex structure) is the same as

\[ M + \left( \text{reduction of the structure group } O(l,m,l,m) \text{ of } M \right) \to U(m,m) \]

Now \( U(m,m) \) is homotopy equivalent to \( U(m) \times U(m) \).

This gives the following picture:
where

\[ V \otimes V^+ = T \otimes T^* \]

with

metric \( |V| = \text{positive definite} \)

metric \( |V^+| = \text{negative definite} \)

Also projections

\[ V \rightarrow T \]
\[ V^+ \rightarrow T \]

are isomorphisms and we get two almost complex structures

\((V, I_+)\)
\((V^+, I_-)\)

In particular: if \( M \) - generalized almost complex \( \mathcal{T} \) has two natural almost complex structures.
Thus spheres can not be generalized almost complex if dim > 2.

Now going back to $I_+ , I_-$ note that $L -$ canonical bundle on $M,$
then

$$ -2 c_1 (L) = c_i^+ + c_i^- $$

Note: symplectic case: $c_i^+, c_i^- -$ opposite,
complex case: $c_i^+, c_i^- -$ same

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**Generalized complex submanifolds**

If $Y \subset M -$ submanifold

$$ TY \subset TM | Y $$

$$ N^*|Y = (TY)^{\text{ann}} \subset T^*M | Y $$

$$ \Rightarrow TY \oplus N^*|Y \subset T \oplus T^* | Y $$

**Def:** A **generalized complex manifold** is a submanifold $Y \subset M$ in a generalized complex manifold $M$ s.t. $TY \oplus N^*|Y$ is preserved by $J.$
Example: 1. \( M \) - complex

\[
J = \begin{pmatrix}
I_T & 0 \\
0 & -I_T^* 
\end{pmatrix}
\]

\( \Rightarrow \) \( Y \subset M \) is a generalized complex submanifold \( \Rightarrow Y \) is a complex manifold

2. B-field transform of a complex manifold:

\[
\begin{pmatrix}
1 & 0 \\
-B & 1 
\end{pmatrix}
\begin{pmatrix}
I_T & 0 \\
0 & -I_T^* 
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
B & 0 
\end{pmatrix}
=
\begin{pmatrix}
I_T & 0 \\
-(B + iB^t) & -I_T^* 
\end{pmatrix}
\]

\( \Rightarrow \) \( Y \subset M \) is a generalized complex

manifold.

\[ B |_Y \] is of type \((1,1)\) (because \( B(x_1, x_2) + B(x_1, i x_2) = 0 \))
(2.) If \( M \) - symplectic, \( Y \subset M \) - submanifold

\[
J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}
\]

\( \omega^{-1} \xi \in T_Y \Rightarrow Y \) - coisotropic.

\( \omega \xi \in (T_Y)^{\text{ann}} \Rightarrow Y \) - isotropic.

(\( \Rightarrow \)) \( Y \) - generalized complex submanifold

(\( \Leftarrow \)) \( Y \) - Lagrangian.

(4.) \( B \) - field deformation of symplectic manifold

\[
J \begin{pmatrix} x \\ \zeta \end{pmatrix} = \begin{pmatrix} -\omega^{-1}(\zeta + Bx) \\ (\omega^{-1} B \omega^{-1} B)x + B \omega^{-1} \zeta \end{pmatrix}
\]

\( Y \subset M \) - generalized complex \( \Leftrightarrow \)

- \( Y \) - coisotropic.

- \( B \) - field is basic, i.e., is a pullback from the leaf space of the null foliation on \( Y \) corresponding to \( \omega \).

\( B + i \omega \) is the \((2,1)\) form of a complex structure on the leaf space.
In other words: \( Y \) is equipped with a transverse holomorphic symplectic structure for the same foliation of \( W \).

Remark: This is almost the same as Kapustin-Olshanskii generalized A-branes.

Only there one requires

\[
B = F_\theta
\]

for some connection \( \theta \) on a line bundle on \( Y \).

- Points are not always generalized complex submanifolds.

In the example with the Poisson structure on a surface

\[ C \]

- Points on \( C \) are generalized complex submanifolds, but points not on \( C \) are not generalized complex submanifolds.