As we explained before: in 6 dimensions the correct group of automorphisms of a generalized complex structure (for the purposes of moduli) was

$$Diffeo M \times \mathbb{R}^2 \text{ exact}$$

This turns out to be the right group in general.

Remark: It is unrealistic to expect that $$\mathbb{R}^2$$ closed will be an appropriate group to use for moduli constructions.

For instance if $$M^{4k}$$ hyper-kähler then

$$(w_1 + iw_2)^{k+1} = 0$$

and if we act by $$e^{\exp \left( \frac{w_1 + iw_2}{t} \right)}$$

we get a B-field transforming to a complex structure

$$e^{\exp \left( \frac{w_1 + iw_2}{t} \right)} \rightarrow (w_1 + iw_2)^k$$

$$\frac{w_1}{t} = B \text{- field}$$

$$\frac{w_2}{t} = \text{symplectic}$$

If we divide by all closed B-fields we will not get convergence.
\[
\exp \left( \frac{\omega}{t} \right) \xrightarrow{t \to 0} 2
\]

if we quotient by discussed we will get a non-Hausdorff space.

Let us study the Kuranishi theory of a generalized complex structure:

\[ E \subset (T \otimes T^*) \otimes C \]

- \( E \) - maximal isotropic
- \( E \cap \overline{E} = 0 \)
- sections of \( E \) are closed under the Courant bracket.

The first obstacle of doing deformation theory here is the fact that the Courant bracket is not a Lie bracket.

If \( \begin{bmatrix} A \end{bmatrix} \in C^0(\mathbb{T} \otimes T^*) \), then we have the Courant Jacobiator:

\[
\begin{bmatrix} [C, A \otimes B], C \end{bmatrix} + \begin{bmatrix} [B, C \otimes A], A \end{bmatrix} + \begin{bmatrix} [C, A \otimes B], B \end{bmatrix} =
\]

\[
= -\frac{1}{3} d \left( \begin{bmatrix} [A, B \otimes C], C \end{bmatrix} + \begin{bmatrix} [B, C \otimes A], A \end{bmatrix} + \begin{bmatrix} [C, A \otimes B], B \end{bmatrix} \right)
\]

Since \( E \subset (T \otimes T^*) \otimes C \) is isotropic it follows that the Jacobiator \( = 0 \) if \( A, B, C \in \mathbb{E} \).

Thus \( E \) together with its natural projection

\[ E \overset{i}{\rightarrow} T \otimes C \]

and the Courant bracket is a Lie algebroid.
Consider how
\[ d : C^0(\Lambda^k E^*) \to C^0(\Lambda^{k+1} E^*) \]
given by
\[
d(\sigma(A_0, \ldots, A_n)) = \sum (-1)^i \bar{\nu}(A_i) \sigma(A_0, \ldots, \hat{A_i}, \ldots, A_n)
\]
\[ + \sum_{i < j} (-1)^{j+i} \sigma([A_i, A_j], A_0, \ldots, \hat{A_i}, \ldots, \hat{A_j}, \ldots, A_n) \]
constant bracket

Since \( E \) satisfies Jacobi on \( E \Rightarrow d^2 = 0 \)
(This in fact works on any Lie algebroid)

Next recall that \( E \) had a complementary subbundle \( \bar{E} : E \cap \bar{E} = 0 \). Since \( E \subset (T \oplus T^*) \otimes C \) was isotropic \( \Rightarrow E = E^\perp \Rightarrow \) (1) induces an isomorphism
\[ \bar{E} \cong E^* \]
(This turns \( E \) into a Lie bialgebroid.)

Infiniteesimal deformations of \( E \subset (T \oplus T^*) \otimes C \); these are given by the tangent space to the Grassmannian:
\[
\text{Hom}(E, (T \oplus T^*) \otimes C / E) = \text{Hom}(E, \bar{E})
\]
\[ = \text{Hom}(E, E^*) = E^* \otimes E^* \]
The infinitesimal deformations of $E$ as an isotropic subspace of $(T \oplus T^*) \oplus \mathbb{C}$ are then given by 

$$\Lambda^2 \mathbb{E}^* \subset \mathbb{E}^* \otimes \mathbb{C}$$

Also we would like to study not just infinitesimal deformations but infinitesimal deformations modulo our symmetries:

$\text{Lie}(\text{Diff} \times \mathbb{R}^2\text{exact})$.

Note: We have a natural complex

$$C^\infty(\mathbb{E}^*) \overset{d}{\to} C^\infty(\Lambda^2 \mathbb{E}^*) \overset{d}{\to} C^\infty(\Lambda^2 \mathbb{E}^*)$$

$$\uparrow$$

$$(T \oplus T^*) \otimes \mathbb{C}$$

Here $$(T \oplus T^*) \otimes \mathbb{C} \overset{\rho^T}{\to} \mathbb{R} \cong \mathbb{E}^*$$ is the natural map

$$x + z \mapsto x + \frac{d}{dx}z$$

$$(\text{Lie Diff} \times \mathbb{R}^2\text{exact}) \overset{\rho^T}{\to} C^\infty(T)$$
Proposition: This is the deformation complex for generalized complex structures.

Examples:
- Symplectic: de Rham complex for complex forms
- Complex: \( \Lambda^0(T \oplus \overline{T} \oplus \overline{T}) \), \( T = \text{holomorphic tangent bundle} \)

\[
\begin{align*}
\overline{T} \oplus \overline{T} & \to \Lambda^2 \overline{T} \oplus \overline{T} \\
T \oplus \overline{T} & \rightarrow \Lambda^2 T \oplus \overline{T}
\end{align*}
\]

Theorem (Gualtieri): There is a Kuranishi space which is smooth and has a tangent space equal to \( H^2(\text{complex}) \).

\[ \partial \sigma + \frac{1}{2} \{ \sigma, \sigma \} = 0 \]

"Schouten bracket"

Examples: Deformations of a complex manifold as a generalized complex manifold

\[ H^2 = H^2(U) \oplus H^1(T) \oplus H^0(\Lambda^2 T) \]

(0,2) part defines a closed B-field structure.
Recall on complex manifolds the holomorphic Poisson structures act on the complex structure to produce a generalized complex structure:

\[ E = \sum \frac{\partial}{\partial z_i} \otimes d\bar{z}_i \]

If \( B = \sum \frac{\partial}{\partial z_i} \otimes \frac{\partial}{\partial \bar{z}_j} \), then

\[ E \rightarrow \sum \left( \frac{\partial}{\partial z_i}, dz_i + \frac{\partial}{\partial \bar{z}_i} \right) \]

sends us to another generalized complex structure.

- If \( \eta^{-1} \) exists so \( \eta^{-1} \) is a holomorphic symplectic form and \( \eta \) acting on \( E \) is the same as the \( \eta^{-1} \)-transform of \( E \) where \( \eta^{-1} \) is considered as a B-field.

- The obstruction space in this case is

\[ H^2 (\text{complex}) = \bigoplus_{\rho \neq q = 2} H^0 (\Lambda^q T) \]
K3 surfaces

If $X$ is a K3 surface, then

$$H^2(\text{complex}) = H^0(T) \oplus H^1(T) \oplus H^0(T^*)$$

The obstruction space is $= 0$.

One can check that near a complex point, every generalized complex structure on a K3 is a generalized CY structure.

This can be made more explicit by taking

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_2 + \mathcal{E}_4$$

pure symmetry describing a generalized complex structure.

Now:

$$\begin{align*}
\mathcal{E}^2 &= 0 \\
\langle \mathcal{E}, \mathcal{E} \rangle &= 0 \\
\langle \mathcal{E}, \overline{\mathcal{E}} \rangle &= 0
\end{align*}$$

are the conditions that $\mathcal{E}$ defines a generalized complex structure.

Consider now the analogous locus in cohomology:

$$\mathcal{E} \in H^2(\mathbb{P}(\mathcal{E}_0, \mathcal{E}_2, \mathcal{E}_4)) : \mathcal{Q} \subset \mathbb{P}(H^2(\mathbb{P}(\mathcal{E}_0, \mathcal{E}_2, \mathcal{E}_4)))$$

an Hilbert discriminant, given by $\langle \mathcal{E}, \overline{\mathcal{E}} \rangle = 0$.
Take an open set of all $\{e_i\} \in \mathcal{Q} : \langle e_i, e_i \rangle > 0$

This is the period space for generalized complex structures on $M$.

We have a period map

$$\phi : \{e_i\} \rightarrow \mathcal{Q}$$

The period space is given by

$$\mathcal{Q} = \frac{SO(4, 20)}{SO(2) \times SO(2, 20)}$$

Can use as a local model for the moduli space.

How about the moduli space of generalized CY metrics?

We need two pure spinors $\psi_1, \psi_2$:

$$\langle \psi_i, \psi_i \rangle = 0 \quad \langle \psi_i, \overline{\psi}_j \rangle = 0$$

If we write

$$\psi_i = x_i + y_i \overline{x_i}$$

$x_1, x_2, y_1, y_2$ are orthogonal w.r.t. Mukai pairing and have unit length.
So the image of the moduli of generalised CY metrics on $M$ in the period space is open in $\text{SU}(4,20) / \text{SU}(4) \times \text{SU}(2) \times \text{SU}(20)$.

Let us now look at the generalised CY metrics on a K3 $\mathcal{M}$ from the point of view of the generalised Kähler metrics.

Recall that a generalised Kähler structure on $M$ is given by a pair of complex structures $J_+$ giving say for $J_+$ a $B$-field automorphism:

$$\mathcal{A} \left( \frac{\partial}{\partial z^i} \right) = \mathbb{R} \left( \frac{\partial}{\partial z^i} \right) + i \mathbb{R} \left( \frac{\partial}{\partial \bar{z}^i} \right) (\mathcal{B} + \mathbb{R} \omega + \bar{\mathbb{R}} \omega)$$

Given a generalised Kähler structure on $M$ we need to find $\mathcal{U}_1, \mathcal{U}_2$:

$$\mathcal{A} \left( \frac{\partial}{\partial z^i} \right) \mathcal{U}_i = 0$$

But

$$\mathcal{A} \left( \frac{\partial}{\partial z^i} \right) \mathcal{U}_1 = 0 \iff i \mathcal{A} \left( \frac{\partial}{\partial z^i} \right) \mathcal{U}_1 = 0 \iff \mathcal{B} + \mathbb{R} \omega + \bar{\mathbb{R}} \omega = 0$$

implies $\mathcal{U}_1, \mathcal{U}_2$ is a $(0,0) + (0,2)$ form.
Here
\[ \psi_1 = e^{-i\omega+b} (\mathbb{I} + \mathbb{X}, \mathbb{Y}) \]
\[ \psi_2 = e^{-i\omega+b} (\mathbb{Z} + \mathbb{Y}, \mathbb{Z}) \]

with \( \mathbb{Z} \mathbb{Y} - \mathbb{Z} \mathbb{X} \neq 0 \)

Impose algebraic conditions \( \Rightarrow \) \( \mathbb{Z} \mathbb{Y} = 0 \)

or \( \mathbb{Z} \mathbb{X} = 0 \)

So
\[ c = \text{constant} \]
\[ \psi_1 = c e^{-i\omega+b} \]
\[ \psi_2 = e^{-i\omega+b} \]

\((0,1)-form\)

So \( d\psi_1 = 0 \) implies \( d(-i\omega+b) = 0 \)
\( \Rightarrow d\omega + db = 0 \)
\( \Rightarrow \nabla^+ = \nabla^- = D - \text{Levi-Civita} \)

\( \equiv M - \text{Kähler} \)

Also \( d\psi_2 = 0 \) implies \( \overline{\omega} = 0 \) \( \Rightarrow \) \( \overline{\omega} \) - holomorphic

Implying \( \langle \psi_1, \overline{\psi}_1 \rangle = \langle \psi_2, \overline{\psi}_2 \rangle \Rightarrow |\phi|^2 = 1 \)

Thus \( \phi \) must be covariantly constant.

Thus we get an ordinary CY metric on the
\( \mathbb{C}^3 \) \( M \) - closed B-field
So the moduli space now becomes

\[ \text{SO}(4,20) \rightarrow \text{SO}(4) \times \text{SO}(20) \]

\[ \text{SO}(4) \times \text{SO}(20) \]

\[ \text{B-fields} \]

\[ \text{moduli of \ he \ metrics \ on \ M} \]

\[ \text{moduli of \ } N = (\mathbb{C}^4, \mathbb{C}^4) \text{ SCFT} \]

How will all this look for higher dimensional

\[ \text{he \ manifolds?} \]

Look at a primitive \ he \ manifold

\[ H^1 \otimes H^1 (T) \otimes H^0 (\Lambda^2 T) \]

Now \ Poisson \ structures \ are \ invertible

\[ \text{in \ acting \ by \ phase = \ taking \ a \ B-field \ transform \ by \ a \ holomorphic \ symplectic \ form.} \]

Now one gets that near a complex structure

\[ \text{on a he \ m \ we \ can \ get \ all \ generalized} \]

\[ \text{complex \ structures \ on \ m \ as \ limits \ of} \]

\[ \text{B-field \ transforms \ of \ complex \ structures.} \]
to get a moduli space contained in
\[ \text{H} \left( H^{00}(M, C) \right) . \]

Verbitsky/Bogomolov: If \( H = H^2 \subset H^{00}(M^{re}, C) \)
then the subalgebra generated by \( H \)
is

\[
\text{Sym} H / \left( \text{ideal generated by } c^{-1} \right) \\
\text{for all } \sigma \in Q(\sigma) = 0
\]

Here \( Q \) is the Beauville-Bogomolov form
on \( H \).

Periods lie in closure of \( c \exp(\beta i \omega) \)
and one expects that there may be
a bigger group acting on \( H \) and
containing \( O(Q) \) so that

\[
\text{closure of } c \exp(\beta i \omega) \subseteq \left( \text{Sym} H / \sigma: O(\sigma) = 0, \sigma = 0 \right)
\]

the spins of the
orbit of the
bigger group