Algebraic topology and differential forms I

Remark: Topological constructions answering questions in physics usually give an answer that is too crude.

Example: If $\Sigma$ is a manifold and $\mathcal{S}$ parameter space and

$$d \in H^3(\Sigma \times \mathcal{S})$$

a family of cohomology classes on $\Sigma$ parameterized by $\mathcal{S}$, then $d$ most naturally gives rise to

$$\Sigma : d \in H^1(\mathcal{S}, 2)$$

i.e. $\Sigma$ is homotopy class of maps

$$\Sigma : \mathcal{S} \rightarrow \mathcal{S}'$$

However in physics this is not enough. We would like to have an actual map rather than a homotopy class.
The goal of the lectures is to describe a refinement of topology that will produce a more "physical" answer.

Based on a joint work with I. Singer and D. Freed.

**Differential functions.**

If \( M \) is a smooth manifold, \( X \) a topological space, \( i \in Z^n(X, \mathbb{R}) \) a cocycle on \( X \), then we define a differential function from \( M \) to \((X, i)\) as a triple:

\[
(c, h, w) : M \rightarrow (X, i)
\]

where

\[
c : M \rightarrow X \quad \text{continuous map}
\]

\[
h \in C^{n-1}(M, \mathbb{R})
\]

\[
\delta h = w - c^*\delta i
\]

\[
w \in \Omega^n(M)
\]
We would like to do algebraic topology starting with differential functions rather than functions.

For this we need to understand how differential functions vary. It is convenient to introduce some simplicial or combinatorial machinery as a tool.

Recall: If $Y$ is a space, then the topological invariants of $Y$ are contained into a sequence of sets and maps between them:

$$\text{Sing}_0(Y) = \text{pts of } Y$$

$$\text{Sing}_1(Y) = \{\text{maps } \Delta^1 \to Y\}$$

$$\text{Sing}_2(Y) = \{\text{maps } \Delta^2 \to Y\}$$

and the maps between $\text{Sing}_i(Y)$ and $\text{Sing}_{i+1}(Y)$ tell us how different simplices are glued to each other.
The collection of sets $S_{ij}$ together with the maps $S_{ij} \rightarrow S_{jk}$ (faces + degeneracies) is what is known as a simplicial set.

Simplicial sets are useful devices and we will use them to make the differential functions into a space.

**Def.** The space of differential functions

$$(X, i)^M$$

is a space with $n$-simplices

$$(c, h, i) : M \times A^n \rightarrow (X, i)$$

Furthermore, this space is filtered and

$$\text{filt}_k (X, i)^M = \text{space whose } n\text{-simplices}$$

$$(c, h, i) : M \times A^n \rightarrow (X, i)$$

with

$$\omega \in \Omega^j(M) \otimes \Omega^j(M)$$

for $j \leq k$. 


Note: We need the filtration as part of the structure. If we forget about the filtration, then the spaces of differential functions will not give anything more than the spaces of ordinary functions.

We would like to understand the space of differential functions and its topological invariants in examples.

Recall: One of the most basic topological invariants of a space is its fundamental group or more generally its fundamental groupoid. (If $Y$ is a space, then the fundamental groupoid $\Pi_1(Y)$ is a category with objects = points of $Y$, maps = paths in $Y$ connecting points, composition law = composition of paths.)
Examples:

- $X = \mathbb{C}P^\infty$

Then for any manifold $M$, we have

$$\pi_1 \text{Hilb}((X, i)^M)$$

is the category with

**Objects:** $U(1)$ bundle on $M$ with connection

If this object is given by

$$(c, b, u) : M \to (X, i)$$

then

$W \in \Omega^2(M)$

is the curvature 2-form of

our connection.
maps \pi \text{ principal bundle maps}
\text{modulo homotopy}

This is equivalent to
\[ \text{Maps} (M, C_p) \]

usual purely topological notion of maps

In other words if we take the second stage in the filtration of the space of differential forms then we recover ordinary topology.

But the data of the filtration carries more info.

If we look at the filters \((X,i)^M\)
we get a groupoid with

objects: \(U(1)\) bundles with connection

maps: Principal bundle maps

with an equivalence on them
2. Review: To see that these are the maps we look at for homotopy paths.

and a disc bounding them

Then the condition that we are in fibras says that

$W$ has at most a 1-form component on $\Delta$

Finally we can look at

the fiber $(X, i) M$

which is now a groupoid with

objects: principal $U(1)$ bundles with

connections

maps: horizontal connection preserving
Note: To see that these are the right maps note that the condition for being in \( \Gamma \Theta \) says that the curvature form in 

\[ \omega \in \mathbb{R}^2(M \times [0,1]) \]

must be constant along \([0,1]\).

This implies that the holonomies of our connection do not change along \([0,1]\).

Generalization: Take \( X = \mathbb{K}(\mathbb{Z}, n) \) and let \( x \in \mathbb{Z}^n(X, \mathbb{R}) \) be a representative of the generator of \( H^n(X, \mathbb{Z}) \).

Now 

\[ \pi_0(\chi^M) = \{ M, X \} = H^n(M, \mathbb{Z}) \]

\[ \pi_0(\chi^M_{\mathfrak{h}, \Theta}) = \text{Cheeger-Simons} \]

\[ \text{Cohomology Group} \]

\[ H^{n-1}(M) \]

\[ = \text{Deligne Cohomology Group} \]

\[ H^{n,n}(M) \]
This group fits in a short exact sequence:

\[ 0 \rightarrow H^{n-1}(M, \mathbb{R}/\mathbb{Z}) \rightarrow H^n(M) \rightarrow \mathbb{C}^\mathbb{R}/\mathbb{Z} \rightarrow 0 \]

Closed forms with integral periods.

Again we see that fiber carries more refined geometric information than the ordinary topological mapping spaces.

In fact we can take \( X = \text{space representing some kind of cohomology theory} \ E \).

For instance, \( X = K(\mathbb{Z}, n) \) represents ordinary cohomology \( H^n \).

\( X \) is space of Fredholm operators = classifying space for K-theory.

Now looking at \( F \) of \( (X, \xi) \) we get a differential version of
The cohomology theory $E$

(We will denote this theory by $E(n)(M)$)

Example: Differential $K$-theory is the $K$-theory of vector bundles with superconnections.

Preliminary description of $H^{n-1}(M) = H_{n-1}(M)$.

- The original Cheeger-Simons definition:

$H^{n-1}(M)$ is pairs $(\xi, W)$ consisting of a homomorphism $\mathbb{Z}_n(M) \to \mathbb{R}/\mathbb{Z}$

$s.t.$

$\chi(\omega W) = \int_{\mathbb{N}} \omega$

Deligne's definition
Consider
\[ Z(n) := Z \rightarrow D^0 \rightarrow D^1 \rightarrow \cdots \rightarrow D^{n-1} \]
de Rham complex stupidly truncated at level \( n-1 \).

Then
\[ H^m(M, Z(n)) := H^m(M, Z(u)) \]
for all \( n \).

A definition coming from the differential forms point of view.

Consider a complex \( C(n)(M) \) given by
\[ C(n)^k(M) := \text{set of } (c, h, \omega) \]
where \( c \in C^k(M, \mathbb{Z}) \),
\( h \in \mathbb{R}^k(M) \)
and \( \omega \in \mathbb{R}^{k+1}(M, \mathbb{R}) \)
with a differential
\[ \delta(c, h, \omega) = (\delta c, \delta h, -\omega + c, d\omega) \]
\( \delta (c, b, w) = 0 \iff c, w \text{ - closed} \)
\[ \delta h = w - c \]

Now \((\text{with cohomology group of } C_\infty(M)) = H^{m,n}(M), \)

The "ordinary" differential cohomology \( H^{m,n}(M) \) as we saw correspond to
the differential function spaces from \( M \) to Eilenberg-Mac Lane
spaces. For these differential cohomologies we have all the usual apparatus of
cohomology (cup product, integration, ...).

Let's illustrate this on an example:

Let \( G \) - compact Lie group
\[ X = BG \]
\[ i \in \mathbb{Z}^\ast (BG, \mathbb{R}) \]

Now given a principal \( G \)-bundle with
connection on \( M \) gives rise (Chern-Weil)
to a differential function \( M \rightarrow (BG, i) \)
(c.f.: If \( G \neq U(1) \), then a differential function \( \mathcal{M} \to (BG, i) \) contains less information than a connection.

Fix a principal \( G \)-bundle \( \mathcal{E} \) on \( \mathcal{M} \) then we get a map

\[
\mathcal{A} \to (BG, i)^\mathcal{M}
\]

Space of connections

In fact this map lands in \( \text{Saltto} \ (BG, i)^\mathcal{M} \)

Let

\[
(c, h, w) : \mathcal{M} \times \mathcal{A} \to (BG, i)
\]

be the corresponding differential function

If then \( \mathcal{M} = \mathbb{Z}, \ G = U(1) \Rightarrow \)

\[
\int (c^*, h, w) = \text{differential function}
\]

\[
\mathcal{A} \to (\text{Coneo}, i) \quad \mathcal{M} = U(1) \text{ bundle with connection on it}
\]
This leads to a cocycle refinement of the classical Chern-Simons theory as worked out by Gawedski.