D-branes, Mirror Symmetry and all that

Lecture 1

Main motivations:
- Homological mirror symmetry conjecture
- Boundary to topological string theory
- Deriving closed string amplitudes from open string amplitudes.

\( \mathcal{X} \) - Calabi-Yau manifold:
1. Complex manifold
2. Of Kähler type
3. \( c_1(\mathcal{X}) = 0 \)

The data we will consider will consist of \( (\mathcal{X}, W, B) \):

- Complex manifold
- Kähler form
- Closed 2-form (B-field)

\( \text{Note: the physics only depends on } [B] \in \frac{\text{H}^2(\mathcal{X}, \mathbb{R})}{\text{H}^2(\mathcal{X}, \mathbb{Z})} \)

Given \((\mathcal{X}, W, B)\) one constructs a \( N=1 \) super conformal field theory. This construction is perturbative in \( \frac{1}{\text{Vol}(\mathcal{X})} \) and is believed to be convergent.
A $N=2$ SCFT is a complicated structure. Roughly it consists of:

- An infinite-dimensional vector space
- An action of 2 copies of $N=2$ super Virasoro algebra.

**Question:** When do we have an isomorphism of $N=2$ super conformal field theories? That is, for what $(X, W, B)$ and $(X', W', B')$, do we have

$$
\begin{align*}
(X, W, B) & \overset{\text{iso}}{\longrightarrow} (X', W', B') \\
\end{align*}
$$

**Note:** If we are only interested in a $N=1$ iso of $N=2$ SCFT, we can require it to induce a flip $J \leftrightarrow Q$ (mirror transformation) on the generators $J_n, L_n, n \in \mathbb{Z}$.

Of our super Virasoro actions...
If we want to flip in both types of supermanifolds then we just get
\[ X \leftrightarrow \hat{X} \]
conjugation.

Or, if we take the identity on one of these and flip on the other we get mirror symmetry.

Now we are searching for pairs \((X, \omega, B)\) and \((\hat{X}, \hat{\omega}, \hat{B})\)
set.

\[
\begin{align*}
(X, \omega, B) & \quad \leftrightarrow \quad (\hat{X}, \hat{\omega}, \hat{B}) \\
N=2 \text{ SCFT} & \quad \leftrightarrow \quad N=2 \text{ SCFT}^\text{mirror}
\end{align*}
\]

**Ansatz:** Instead of \(N=2\ \text{SCFT}\), let's look at its topologically twisted version:

\[
\begin{align*}
N=2 \text{ SCFT} & \quad + \text{ Integrality conditions} \quad \rightarrow \\
\text{A-model} & \quad \leftrightarrow \quad \text{B-model}
\end{align*}
\]
1. Biological twist:

- Redefine spins of all fields: the A becomes
  \[ q = 0 \]
  \[ h' = h + \frac{q}{2} \]
  \[ \tilde{h}' = h - \frac{q}{2} \]

- BRST operator

(need redefined spins to be integral for all fields)

\[ h' = h - \frac{q}{2} \]
\[ \tilde{h}' = h - \frac{q}{2} \]

For SUSY \( \sigma \)-model, the integrality always works but for LG models

only \( B \)-twist makes sense. Similarly

if \( X \) is not CY \( \Rightarrow \) \( N=2 \) non-conformal

\( \sigma \)-model \( \Rightarrow \) can only do \( A \)-twist
The A/B models can be rigorously defined:

Topological Field Theory: graded superconformal algebra with
\[ \langle u, v \rangle = \langle u v, w \rangle \]
i.e. a superconformal Frobenius algebra

For the A/B models, these algebras are:

A-model: \( \bigoplus H^q(X) \) + quantum cohomology
\[ \text{trace} = \int_{\Sigma} \]

B-model: \( \bigoplus H^p(\Lambda^q TX) \) + usual product
\[ \text{holomorphic} \]
\[ \text{volume form} \]
However if we want to understand more of $\mathcal{N}=2$ SCFT we need either structure: replace algebras by categories (of topological D-branes).

Now if we forget the B-field

$A$-model depends only on the symplectic structure

$B$-model depends only on the complex structure

Kontsevich:

$A$-model: derived Fukaya category (A-type D-branes)

$B$-model: $D^b(X)$ - derived category of coherent sheaves

Objects should be thought of as D-branes and morphisms as strings stretching between two branes.
Description of $D^b(X)$:

Consider the category of $(E, \mathcal{F})$,

$E$ complex vector bundle

$\mathcal{F} : E \otimes \Omega^{1,r} \rightarrow E \otimes \Omega^{1,r+1}$

This is the same as the category of holomorphic vector bundles on $X$.

One needs to repair this category to make it abelian: consider the category $\text{Coh}(X)$ obtained from the category of holomorphic vector bundles by adding formally the object $(E \otimes \mathcal{F})$ to the objects.

The category $\text{Coh}(X)$ is not the right one for describing D-branes.

If $X =$ complex torus and $\hat{X} =$ dual torus

$\Rightarrow \text{Coh}(X) \neq \text{Coh}(\hat{X})$ whereas T-duality predicts that the corresponding NS5 SCFT's are the same.
To fix this: replace \( \text{Coh}(X) \) by \( \mathcal{D}^b(X) \) - derived category of coherent sheaves.

Now \( \mathcal{D}^b(X) \cong \mathcal{D}^b(\hat{X}) \) via Fourier-Mukai transform.

The objects of \( \mathcal{D}^b(X) \) are complexes

\[
\ldots \rightarrow A_1 \xrightarrow{d_1} A_2 \xrightarrow{d_2} A_3 \rightarrow \ldots
\]

of coherent sheaves.

The maps between such objects are built in several steps:

- **Step 1:** Take as a first approximation to the space of morphisms maps.

\[
\ldots \rightarrow A_1 \xrightarrow{d_1} A_2 \xrightarrow{d_2} A_3 \rightarrow \ldots
\]

\[
\xrightarrow{\xi_1} \xrightarrow{d_1 e_2} \xrightarrow{d_2 e_3} \xrightarrow{d_3}
\]

\[
\rightarrow B_1 \xrightarrow{d_1} B_2 \xrightarrow{d_2} B_3 \rightarrow \ldots
\]
So that all the squares commute. In physical terms:

\[
(A_i, d_i) \leftrightarrow A = \bigoplus A_i
\]

\[
d = \bigoplus d_i, \quad d : A \to A, \quad d^2 = 0
\]

Then \( \epsilon : A \to \mathbb{B} \) is just such that

\[
[d_1, \epsilon] = 0
\]

We should quotient the space of such \( \epsilon \)'s by homotopies:

\[
\begin{array}{c}
A_1 \xrightarrow{d_1} A_2 \xrightarrow{d_2} A_3 \\
\downarrow \epsilon_1 \downarrow \epsilon_2 \downarrow \epsilon_3
\end{array}
\]

\[
\begin{array}{c}
B_1 \xrightarrow{\epsilon_1} B_2 \xrightarrow{\epsilon_2} B_3
\end{array}
\]

(9843.1235678.321) \( \mathcal{M} = \{ \epsilon \mid d_1 \epsilon = \epsilon d_2 \} \)

\[
\epsilon \sim \epsilon' + [d_1, \epsilon]
\]

Step 1: Invert formally the \( \epsilon \)'s which are quasi-isomorphisms. i.e. \( \epsilon : A \to \mathbb{B} \) which induce an isomorphism on cohomology.
Physically: One should enhance morphisms in the derived category $D^b(X)$ to graded vector spaces:

$$\bigoplus \text{Hom}(A, B[k])$$

Here $[k]$ is the shift to the left $k$ steps.

Sometimes one writes:

$$\text{Ext}^k(A, B)$$

for $\text{Hom}(A, B[k])$.

Note: If $E, F$ are vector bundles

$$\text{Hom}(E, F[k]) = H^k(X, \text{Ext}^0(E, F))$$

Fukaya category:

Let $X$ be a symplectic manifold with $\omega(X) = 0$. Consider $M_{00}(n) = \text{universal cover}$ of $S^0(n)$:

$$0 \to \mathbb{Z} \to M_{00}(n) \to S^0(n) \to 0.$$
One needs $X$ to be a metaplectic manifold, i.e., one needs the spin frame bundle of $X$ to be liftable to a principal $Mp(m)$ bundle.

In fact we need to equip $X$ with such a metaplectic structure.

Note: The obstruction to choosing a $Mp(m)$ structure on $X$ is in $H^2(X,\mathbb{Z})$. It is exactly $c_1(X)$.

There is a similar lifting problem for the bundle of Lagrangian Grassmannians on $X$:

If $\text{Lag}(X) \to X$ - the bundle with $\text{Lag}(X)_x = \text{Grassmannian of Lagrangian subspaces in } T_x X$,

then we can try to lift $\text{Lag}(X)$ to a bundle

$\tilde{\text{Lag}}(X) \to X$ of universal covers of Lagrangian Grassmannians.
(Note:  \( \pi_1(\text{Log}(X)) = \mathbb{Z} \))

It turns out that when \( c_1(X) = 0 \),
we can always do that.

The objects in the Fukaya category
are triples

\[(Y, E, D)\]

so that

- \( Y \subset X \) - Lagrangian submanifold
- \( E \) - Hermitian vector bundle
- \( D^2 = 0 \)

However, \( Y \) cannot be arbitrary.

\[ p \sim \text{End}(p) \quad \text{and} \quad p \text{End}(p) \]
gives a section in \( \text{Log}(X) \).

It turns out that \( Y \) has to be such that the corresponding section
should be liftable to a section

\[ \text{in} \quad \text{Log}(X) \]
In general, there is an obstruction to doing that $e \in \Omega^1(Y, \mathbb{Z})$.

This obstruction is the \textit{Maslov class}.

Explicitly, we have $\varphi \in H^{1,0}(\mathcal{X})$

$$\varphi \mid_Y = f \text{ Vol } Y, \quad f \text{-function on } Y$$

The Maslov class $\varphi$ is given by

$$\varphi = d \log f = \frac{1}{f} df$$

This class is the obstruction for lifting our section. It is also the obstruction to the existence of a single valued branch of $\log f$ on $Y$.

The choice of such a \textit{branch is equivalent to choosing a formal \textit{lift} of the section}.

$Y + \{\text{branch } \log f\}$ will be called a framed Lagrangian submanifold of $Y$. 

framing of $Y$
The morphisms in the category are more complicated.

\[ V = \bigoplus_p \text{Hom}(E_p, F_p) \]

\[ \begin{align*}
\text{grade} &= \text{relative Maslov index} \\
\delta_{p,q} : E_p &\otimes F_p \rightarrow E_q \otimes F_q \\
\delta_{p,q} &= \sum \text{hole maps from } D^2 \rightarrow X
\end{align*} \]
One can define a floor category by taking the spaces of morphisms to be the cohomology of these complexes.

To get the derived Fukaya category one steps back and considers the floor complex as the space of morphisms.

The problem is that composition on the level of floor complexes is not associative. It is only associative up to homotopy — this gives the structure of an A∞ category.

Kontsevich gave a purely algebraic construction of a derived category of an A∞ category. Applying this to the A∞ Fukaya category one gets the derived Fukaya category which enters Kontsevich’s conjecture.