Dr. Anton Kapustin, Caltech (KITP Geometry 7-30-03) Mirror Symmetry, Derived Categories, and D-Branes III

Lecture 3

**X** - complex manifold

**U : X → C** - holomorphic function

**LG model on** (X, W) : today X = C^n

We would like to find the right boundary conditions.

Try Neumann boundary conditions on φ : \( \Sigma \rightarrow X \):

\[
\begin{align*}
&\gamma_0 \phi |_{\Sigma} = 0 \\
&\gamma_4 - \gamma_4 |_{\Sigma} = 0 - \text{fermionic boundary condition}
\end{align*}
\]

\[
\delta S_{\text{bulk}} = \int_{\Sigma} \frac{i}{2} E \left( \psi \bar{\psi} \bar{\psi} W + \psi \bar{\psi} \bar{\psi} W \right) + \text{(conjugate)}
\]

So if \( \partial_i W \neq 0 \) Neumann boundary conditions brane \( N = \Sigma \) susy.

N. Warner, Hosni-Iqbal-Vafa : add Dirichlet boundary conditions in some directions in order to restore susy.
This approach does not always work. Instead, try to restore duality by adding a boundary term to the action.

Consider

\[ S_{\text{boundary}} = \frac{1}{2} \int d\tau d\theta \left( \bar{\Gamma} D_0 \Gamma + \Gamma T + \bar{T} \cdot \bar{\Gamma} \right) \]

where:

- \( \Gamma (\tau, \theta) \) is some new superfield living on the boundary

  (If we have a brane-antibrane configuration \( E_0, E_1 \rightarrow \Gamma \in \text{Hom} (E_0, E_1) \))

  \[ \Gamma (\tau, \theta) = \eta (\tau) + i \theta L (\tau) \]

  \( \eta (\tau), L (\tau) \in \text{Hom} (E_0, E_1) \)

  (will assume \( E_0, E_1 \) unitary line bundles)

- \( T = T (\bar{\Phi}^I (\bar{\tau}, \theta)) \in \text{Hom} (E_0, E_1) \)

  tachyon field

  \[ \bar{\Phi}^I (\bar{\tau}, \theta) = \phi^I + \theta \psi^I \]
In terms of $q, p, T$ we have

$$S_{\text{boundary}} = \frac{1}{2} \int \bar{e} q \, dq + \frac{i}{2} \left( \frac{i}{2} \bar{e} T \right) + \text{hermitian conjugate}$$

$$- \frac{1}{2} \bar{T} T.$$

We need to check that $q, p, T$ can be chosen so that the $\mathcal{N}=2$ susy variation

$$\delta S_{\text{bulk}} + \delta S_{\text{boundary}}$$

is zero.

Analyzing this we see that we must have

$$\delta q = \bar{e} s_1 + \bar{e} s_2$$

$$s_1 = \bar{e} F^+$$

$$s_2 = -\bar{e} \bar{g}$$

$$T = F + G^+$$

with

$$F \in \text{Hom}(E_0, E_1)$$

$$G \in \text{Hom}(E_1, E_0)$$

and

$$\delta S_{\text{boundary}} = \frac{1}{2} \int \bar{e} \left( \bar{\psi} i \bar{\omega} (F^+ G^+) - \bar{\psi} i \bar{\omega} (F G) \right)$$

+ (hermitian conjugate)
and one gets

\[ F_G = G, \quad F = W + \text{const.} \]

**Remark:** With more one can check that this is the right condition even when \( E_0, E_1 \) have higher rank.

This is worked out in papers by Kapustin-Li.

To understand what the categories of branes are we need to look at the boundary piece of the BRST operator

\[ \Omega_\text{boundary} = -iF_G + iG \]

where \( \zeta, \bar{\zeta} = 1 \).

It is convenient to write

\[ \zeta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \bar{\zeta} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]

and so

\[ \Omega_\text{boundary} = \begin{pmatrix} 0 & iG \\ -iF & 0 \end{pmatrix} \]

In the bulk we have:
\[ Q_{\text{bulk}} = \int d\sigma \left( \partial_i \bar{\partial} \left( \psi_+ + \psi_- \right) \phi^i + \left( \psi_+ - \psi_- \right) \partial_i \phi^i \right) \]

We have

\[ Q_{\text{bulk}} = -i \, W \epsilon \Sigma \]

\[ Q_{\text{boundary}} = i \, W \epsilon \Sigma \]

And so \((Q_{\text{bulk}} + Q_{\text{boundary}})^2 = 0\).

**Note:** To see that \(Q_{\text{bulk}}\) anti-commutes with \(Q_{\text{boundary}}\), one notices that in the zero-mode approximation that \(Q_{\text{boundary}} = \) holomorphic \(c\) expression

\[ Q_{\text{bulk}} = \mathcal{F}. \]

To work out the cohomology of \(Q_{\text{bulk}} + Q_{\text{boundary}}\), we will work in the zero-mode approximation. This should be enough for the topological B-theory.

We have:

\[ \delta \psi^i = 0 \quad \delta \bar{\psi}^i = -\bar{\epsilon} \, \psi^i \]

\[ \delta \psi^i = -2i \, \bar{\epsilon} \phi^i \quad \delta \psi^i = 0 \]
\[ C^{\frac{1}{2}} = \psi^n \cdots \psi^1 f^n \cdots f_1 \]

\[ \{ Q_{\text{bulk}}, C_{\frac{1}{2}} \} = C_{\frac{1}{2}} \]

For the boundary BRST operator:

\[ f \in \text{Hom}(E_0 \otimes E_1, E_0 \otimes E_1) \]

\[ \Rightarrow f - \text{matrix} \quad \text{and} \quad \{ Q_{\text{boundary}}, C_{\frac{1}{2}} \} = C_{\frac{1}{2}} \{ Q_{\text{boundary}}, f \} \]

Hence our states are described by (in the zero-mode approximation)

\[ f \in \oplus L^{0,p}(X) \otimes \text{End}(E_0 \otimes E_1) \]

with

\[ Q f = \bar{\partial} f + \{ Q_{\text{boundary}}, f \} \]

Note: \[ \{ \bar{\partial}, Q_{\text{boundary}} \} = 0 \]

and \[ Q_{\text{boundary}} \in \text{center}(E_0 \otimes E_1) \]

\[ \Rightarrow \{ Q_{\text{boundary}}, f \} = 0 \quad \forall f. \]
We can simplify things further since we assumed $X = \mathbb{C}^n$.

If $f = f_0 + f_1 + \cdots + f_r$

\[ f_i : \mathcal{O} \otimes \text{End}(E_0 \otimes E_1) \]

Then $\overline{f} + [\Omega_{\text{boundary}}, f] = 0 \Rightarrow f_r = \overline{f} p_{r-1}

Redefine $f$ as

\[ f' = f' = (\overline{f} + [\Omega_{\text{boundary}}, f]) p_{r-1} \]

\[ f' \sim f \quad \text{cohomologous and} \]

\[ f' = f'_0 + \cdots + f'_{r-1} \]

Continuing this way we can replace $f$ by a $0$-form cohomologous to $f$.

Thus we may assume $f = f_0$

\[ \overline{f}_0 = 0 \]

\[ [\Omega_{\text{boundary}}, f_0] = 0 \]

\[ f : (A, B, C) \]

\[ A \in \text{End}(E_0) - \text{holomorphic endomorphisms} \]

\[ B \in \text{End}(E_1) \]

\[ B \in \text{Hom}(E_0, E_1) - \text{holomorphic homomorphisms} \]

\[ C \in \text{Hom}(E, E_0) \]
Remark: It is natural to consider $A, B, C$ as even and $B, C$ as odd since

$$f = x_0 + 2x_1 + 3x_2 + 4x_3$$

Now the differential $Q$ becomes

$$f \mapsto \begin{bmatrix} Q & \text{boundary} \\ f & J \end{bmatrix}$$

$$\begin{bmatrix} (0, G) & (A, B) \\ (F, O) & (C, D) \end{bmatrix} - \begin{bmatrix} (A - B, G) \\ (-C, D) \end{bmatrix}$$

Hence the category of $B$-branes for $(X, W)$ has hom spaces

$$\{ \begin{bmatrix} (A, B) \\ (C, D) \end{bmatrix} \mid \begin{bmatrix} (0, G) & (A, B) \\ (F, O) & (C, D) \end{bmatrix} - \begin{bmatrix} (A - B, G) \\ (-C, D) \end{bmatrix} \}$$

More explicitly:

branes: $E_0 \leftrightarrow E_1$

$F \circ G = W \cdot \tilde{c}_1 E_0 + \text{const}$

$G \circ F = W \cdot \tilde{c}_1 E_0 + \text{const}$

morphisms: $\begin{bmatrix} G(A, B) \\ C & D \end{bmatrix} : E_0 \otimes E_1 \to E_0' \otimes E_1'$

$[Q \otimes \text{boundary}, \circ] = \text{exact}$
Remarks: This category was proposed by Kontsevich as a category of B-branes in LG models and we just got a physical derivation of this proposal.

If we keep the dg category rather than the cohomology category to get more information. For instance, higher products in the dg-category capture $\eta$-pt correlators in the string field theory.

Examples:

\[ W = z^n \quad \text{on} \quad X = C \]
\[ F = z^{n-k}, \quad G = z^k \]
\[ E = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad E_0 = C \alpha \]
\[ E_1 = C \beta \]

\[ D = [ \partial \text{boundary}, \circ ] \]
\[ Dz = \begin{pmatrix} (Cz^k + Bz^{n-k}) & (D - A)z^k \\ (A - D)z^{n-k} & Bz^{n-k} + Cz^k \end{pmatrix} \]
\[ \ker D = \left\{ \begin{pmatrix} A & B \\ -Bz^{n-k} & A \end{pmatrix} \right\} \]

Modifying by $D$-exact terms gives
\[ B \sim B + \delta z^k \]
\[ A \sim A + \delta z^k \]
Hence

\[
\text{End} \left( E_0 \xrightarrow{F} E_1 \right) = \frac{C[2]}{z^k} \oplus \frac{C[2]}{z^{2k}}
\]

\[
= \langle 1, 0, 0 \mid a^k = 0, \theta^2 = -a^{2k} \rangle
\]

where

\[
\theta = \begin{pmatrix}
2 & a \\
0 & \theta + i z
\end{pmatrix}
\]

\[
\Theta = \begin{pmatrix}
0 & 1 \\
-\theta^{-1} & 0
\end{pmatrix}
\]

Interesting special cases:

* \( n = 2 \), \( w = z^2 \), \( F = i \), \( G = \mathbb{Z} \)

\[
\text{End} \left( C_2 \xrightarrow{F} C_2 \right) = \langle 1, \Phi \mid \theta^2 = -1 \rangle
\]

The appearance of a Clifford algebra here is a feature of any quadratic superpotential.

Let \( X = \mathbb{C}^n \), \( W = \mathbb{Q} Q(x, x) \)

\( Q \) - non-degenerate quadratic form

\[
Q^2 \text{boundary} = W \Rightarrow Q \text{boundary} = \Gamma_1, z^4 + \ldots
\]

\[
\frac{1}{2} \Gamma_i, \Gamma_j = Q_{ij}
\]

\( \Rightarrow \) Clifford module
If $W$ - superpotential with non-degenerate critical pts on $\mathbb{C}^n$, then we get that the category of branes breaks into pieces labeled by the critical pts where each piece is the category of Clifford modules for the Clifford algebra corresponding to the quadratic part of $W$ at the corresponding critical pts.

**Note:** Knörrer periodicity: if

$$W' = W(x) + y_1^2 + y_2^2$$

then

$$\mathcal{D}^b(\text{B-branes } W) \cong \mathcal{D}^b(\text{B-branes } W')$$

This was also recently proved by Orlov in the context of LG models.

**Question:** What happens if we add just one square?

**Proposal:** Consider the category of pairs

$$\left( \begin{array}{c} F_0 \\ g \end{array} \right) \xleftarrow{F} F_1, \text{ B t End}(E_0 \otimes E_1)$$

\text{odd } \text{endo morphism}

This is the category of branes equipped with brane - anti-brane symmetry.
\[ D^b(B_{\text{Branes}} \omega) \cong D^b(B = \text{branes} + \text{boundary}) \]

Note: This was checked for \( \omega = 2a_1 \), \( \omega = \Sigma \text{generators} \).