Transport coefficients and spectral functions from the lattice?

Gert Aarts
(OSU)

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with Jose Maria Marique Resco
JHEP

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Transport coefficients

- RHIC: non-ideal hydrodynamics
  (ideal hydro works extremely well...)
- field theory calculation: highly nontrivial
  what has been computed?
  (using Kubo relations and diagrams)
- QCD: color conductivity
  \[ \rho_{\alpha\beta} \]
  (Marcos Roero and Vassil Bbagott)
- other transport coefficients in gauge theory
  (viscosity, electrical conductivity, flavor diffusion)
  computed using kinetic theory
  (Arnold, Niwa, Witten)

Fully nonperturbative computation of transport coefficients with lattice QCD?

- first attempt: Karch & Wyld (1983)
- continued: Nakamura et al. (1996-...)

How: Kubo relation: shear viscosity

\[ \eta = \frac{1}{20} \lim_{\omega \to 0} \frac{1}{\omega} \int \text{d}^4x \ e^{i\omega t} \langle [\Pi^y(x, t), \Pi^y(x, 0)] \rangle \]

with \[ \Pi^y = \Pi^y - \frac{1}{2} \delta y_4 T_{\text{L}} \]

energy-momentum tensor

spectral function:

\[ \rho_{\alpha\beta}(x-y) = \langle [\Pi^y(x), \Pi^y(y)] \rangle \]

\[ \rho_{\alpha\beta}(\omega, \mathbf{p}) = \int \text{d}^4x \ e^{i\mathbf{p}.\mathbf{x}} \rho_{\alpha\beta}(x) \]

\[ \Rightarrow \eta = \left. \frac{1}{20} \frac{\partial}{\partial \omega} \rho_{\alpha\beta}(\omega) \right|_{\omega=0} \]

relation with euclidean correlator:

\[ G^y_{\alpha\beta}(\tau) = \int \text{d}^4x \langle [\Pi^y(x, \tau), \Pi^y(x, 0)] \rangle \]

\[ \alpha < \tau < 1/4 \]

\[ = T \sum_n e^{i\pi\tau} G^y_{\alpha\beta}(\omega) \]
Dispersion relation:

\[ G_\tau^{\tau}(\omega) = \int \frac{du}{2\pi} \frac{\rho^{\tau}(u)}{\Delta(u)} \]

\[ G_\tau^{\tau}(\tau) = \int_0^\tau du \ k(\tau,u) \rho^{\tau}(u) \]

with kernel:

\[ k(\tau,u) = \frac{\Delta(u)}{\tau + u} \]

\[ = e^{\omega_0 u} \eta(u) + e^{\omega_0 \tau} [1 + \eta(u)] \]

Lattice program:
- compute \( G_\tau^{\tau}(\tau) \) numerically
- reconstruct \( \rho^{\tau}(u) \) from integral equation
  - using ansatz for \( \rho^{\tau}(u) \)
  - using Maximal Entropy Method
- find \( \eta \approx \frac{\partial}{\partial u} \rho^{\tau}(u) \mid_{u=0} \)

Obvious questions:
- \( \rho^{\tau}(u) \) at high \( T \) ?
- \( G_\tau^{\tau}(\tau) \) at high \( T \) ?
- how does \( \eta \) or more generally \( \rho^{\tau}(u) \) at \( u < \tau \)
  manifest itself in \( G_\tau^{\tau}(\tau) \) ?

**How does \( \eta \), or more generally \( \rho^{\tau}(u) \) at \( u < \tau \), manifest itself in \( G_\tau^{\tau}(\tau) \)?**

**Easy:**

\[ G_\tau^{\tau}(\tau) = \int \frac{du}{2\pi} k(\tau,u) \rho^{\tau}(u) \]

\[ k(\tau,u) = \eta(u) e^{-\omega_0 \tau} + \lambda_1(u) e^{-\omega_0 \tau} \]

\( \omega_0 \tau < \tau \):

\[ k(\tau,u) = \frac{2T}{\tau} + O(1/T) \]

\( \tau \)-independent \( \tau \)-dependence

**\( \rho^{\tau}(u) \approx \frac{2T}{\tau} \rho^{\tau}(u) \)**

\( \lambda_1 < \tau \)

Constant contribution determined by integral of \( \rho^{\tau}(u)/u \)

Low-frequency region vs soft dynamics

- transport coefficient

\[ \eta \left( \frac{\partial}{\partial u} \rho^{\tau}(u) \right) \mid_{u=0} \]
\[ \rho_{\pi\pi}(\omega) \text{ at high } T ? \]

\[ \lambda \phi : \quad \Pi_{ij} = \delta_{ij} \phi + \frac{1}{2} \delta_{ij} \phi \quad \text{and} \quad \rho_{\pi\pi}(x-y) = \langle [\Pi_{ij}(x), \Pi_{ij}(y)] \rangle \]

\[ \Rightarrow \text{one-loop expression} : \quad \rho_{\pi\pi}(\omega) = \frac{g^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\omega}{2k^0} \rho(k^0, \vec{k}) \left[ \rho(k^0, \vec{k}) - \rho(k^0, \vec{k} - \hat{\mathbf{\pi}}) \right] \]

with one-particle spectral function \( \rho(x-y) = \langle \phi^\dagger \phi \rangle \)

- **Quasi-particles:**
  - HTL plasmon mass \( m^2 = \frac{\lambda^2 T^4}{24k} + ... \)
  - Finite width \( \Gamma_k = - i \text{Im} \Sigma_k (\omega < 0) \)

\[ \Gamma_k = \frac{2 \pi T^4}{1536 \pi} \]

- **Finite width essential when \( \omega \to 0 \):**
  - Pinch singularities

\[ \rho_{\pi\pi}(\omega) \sim \frac{g^2}{2} \frac{T^4}{\omega} \quad (\omega \to 0) \]

\[ \Gamma \sim \frac{T^4}{\Delta} \sim \frac{T^4}{\lambda^2 \pi} \]

- Ladder diagrams contribute at same order (Jean)

\[ \text{each additional rung } \sim \frac{L^2 T^2}{\Delta} \sim 1 \]

**But:** interested in all \( \omega \)'s

- \( \omega \gg \delta \) no pinch singularity, HTL calculation sufficient
- \( \omega \gg T \) bare calculation

- \( \omega \gg \delta \) one-particle spectral function \( \rho(p) = 2 \pi T^2 (p^0 - \omega)^3 \)

\[ \Rightarrow \rho_{\pi\pi}(\omega) = \Theta(\omega - 2m) \left( \frac{\omega - 2m}{4\pi \rho_0} \right) \frac{\omega}{4\rho_0} \]

- Threshold from HTL
- Large frequencies: \( \rho_{\pi\pi} \approx \frac{m^4}{96\pi} \) zero temperature decay

- \( \omega \approx m \) pinching poles

**One-particle spectral function**

\[ \rho_{\pi\pi}(p) = \frac{1}{2 \pi} \int \frac{dp^0}{\sqrt{(p^0 - \omega)^2 + \delta^2}} \left( \frac{2 \pi T^2}{(p^0 - \omega)^3 + \delta^2} \right) \]

- Poles at \( p^0 = \pm (\sqrt{\omega^2 + \delta^2}) \)

\[ \Rightarrow \rho_{\pi\pi}(\omega) = -\frac{g^2}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{\omega_k^2} \right] \left( n'(\omega_k) \frac{\omega_k}{\omega^2 + \delta_k^2} \right) \]

- Anticipated expression:
  - Distance between the poles \( \omega^2 + \delta_k^2 \)
  - \( \omega \to 0 \): \( \rho_{\pi\pi}(\omega) \sim \frac{\omega}{\delta_k} \)
\[ \rho_{\pi^+}(\omega) = c_1(\omega^2 - \omega_k^2)^{5/2} \left[ \eta(\omega_k) + \frac{1}{5} \right] \]

- \omega \text{ sufficiently large}
- no pinch singularity

\[ \rho_{\pi^+}(\omega) = -\frac{3}{2} \sqrt{\frac{d}{d \omega}} \frac{1}{\omega^{5/2}} \frac{n(\omega)}{\omega} \frac{\omega_k}{\omega^2 + \omega_k^2} \]

with width \( \delta_{\omega} = \frac{1}{\omega_k} B(\frac{m}{T}, \frac{m}{T}) \) (Wang & Heinz)

\[ \rho_{\pi^+}(\omega) \sim \frac{\omega^2}{\omega_k^2} \quad (\omega \gg \omega_k, B=1) \]

Viscosity: \( \sim \frac{d}{d \omega} \rho_{\pi^+}(\omega) \bigg|_{\omega=0} \)
**SU(N_c) gauge theory: same story, more involved**

- $T_{ij} = F_{i}^{a} F_{j}^{a} - \frac{1}{3} \delta_{ij} F^{a} F^{a}$

- $D_{\mu \nu} = P_{\mu \nu} A_{T} + P_{\mu \nu} A_{L}$

  with

  $A_{T}(p) = -\int \frac{d^{4}u}{(2\pi)^{4}} \frac{T_{T}(u, p)}{p^{2} - m^{2}}$

  $A_{L}(p) = \frac{i}{8\pi^{2}} \int \frac{d^{4}u}{(2\pi)^{4}} \frac{T_{L}(u, p)}{p^{2} - m^{2}}$

  $\Rightarrow \rho_{\pi}(\omega) \approx \frac{2}{3} (N_{c}^{-1}) \int \frac{d^{4}u}{(2\pi)^{4}} \left[ n(u) - n(u + \omega) \right]$

- $\omega \ll T$: both contributions match parametrically

- $\rho_{\pi}(\omega) / T^{4} \sim \omega^{3/2}$

- Higher loops

  $\omega \lesssim T$: pinching poles from ladders change details, not characteristic shape

  $3 \lesssim \omega \lesssim T$: three-loop diagram contributes as well

- $\omega \sim \sqrt{T}$ pole-cut contribution dominates

- $\omega \approx T$: smaller frequencies: pinching poles from transverse gluons dominate
Smaller frequencies: pinch singularity

- Use finite gluon damping rate \( \gamma = \frac{q^2 N_c \tau}{4 \pi} \ln \frac{1}{g} \)

[... gauge invariance & ladder diagrams...]

- 1-loop viscosity \( \Pi_{1\text{loop}} = \left( \frac{N_c^2-1}{N_c^2} \right) \frac{\tau^3}{\tilde{g}^2 \ln \frac{1}{g}} \)

  (Pisarski)

  Kinetic theory predicts: \( \Pi = \frac{N_c^2-1}{N_c^2} \frac{\tau^3}{\tilde{g}^2 \ln \frac{1}{g}} \)

  (AMY)

  Contribution from ladder larger than 1 loop result

  Gauge invariant summation of subclass of diagrams

  Unsolved problem

- However, does not affect the characteristic shape of \( G_{\pi\pi} \) much

  Only here

  and does not affect the euclidean correlator

Euclidean correlator

\[
G_{\pi\pi}(\tau) = \int \frac{dw}{2\pi} K(\tau, w) R_{\pi\pi}(w)
\]

- Split contributions

- High:

\[
G_{\pi\pi}(\tau) = \frac{\pi^2 T^5}{3 \sin^2 \theta_u} \left[ (\pi u) (\cos u + \cos 2u) + 6 \sin u + 2 \sin 2u \right] + O(\mu^2 T^4)
\]

  \( u = 2\pi \tau T \)

\[
= \frac{1}{8 \pi^2} \left[ \frac{1}{\tau^5} + \frac{1}{(\sqrt{2} \tau T)^5} + \frac{1}{(\sqrt{2} \tau T + 2)^5} + \frac{2}{(\sqrt{2} \tau T + 2)^5} \right]
\]

Maxwell-Boltzmann statistics, \( n(u) \sim e^{-\gamma/\tau} \)

- Simple result: \( \sim \frac{1}{\tau^5} + \text{reflection symmetry } \tau \rightarrow \frac{1}{\sqrt{2} \tau T} \)

- Central value:

\[
G_{\pi\pi}(\tau = \frac{1}{2\pi T}) = \frac{4 \pi^2}{\sqrt{5}} \left( 1 - \frac{25 \pi^2}{16} \right)
\]

\[
= \frac{4 \pi^2}{\sqrt{5}} T^5 \left( 1 + O(\frac{\mu^2}{T^4}) \right)
\]

- Low: \( \omega \ll T \) expand kernel

\[
k(\tau, w) \approx 2\omega^2 w \int dw \rho(\omega) \omega
\]

\[
G_{\pi\pi}(\tau) = -\frac{2}{3} \int \frac{d^3 k}{(2\pi)^3} \int n'(w_1) \int dw_2 \frac{T^5}{2\pi} \frac{\omega_1}{\omega_1 + 2\omega_2} \frac{\omega_2}{\omega_2 + \omega_3}
\]

\[
= \frac{4 \pi^2}{\sqrt{5}} T^5 \left( 1 + O(\frac{\mu^2}{T^4}) \right)
\]
Euclidean correlator:

\[ G(\tau) = \frac{1}{8\pi^2} \left[ \frac{1}{\tau^5} + \frac{1}{(\pi T - \tau)^5} + \frac{2}{(3\pi T - \tau)^5} + \frac{2}{(\pi T + \tau)^5} \right] + \frac{4\pi^2}{45} \tau^{-3} \]

\[ \text{contribution from interesting low frequencies} \]

Finding:

- spectral function has a characteristic bump shape
- fit used in the literature completely wrong
- fit proposed a better one
- dependence dominated by non-interacting high frequencies
- small frequency part gives independent contribution
- transport coefficients
- lattice calculation in a weakly coupled theory
- non-perturbative lattice calculation
Relevance for other correlators:

Thermal dilepton rate

\[ \text{Im } \Pi^{\mu}_{\nu}(\omega, q) \quad \text{photon polarization} \]

in terms of spectral function:

\[ \rho_{\nu}(x-y) = \langle [f^{\mu}(x), f^{\nu}(y)] \rangle \]

recent lattice calculation using Maximal Entropy Method

(Karsch et al.)

- pinch singularities
- complicated behaviour at very small \( \omega < T \)
- difficult to analyse from the lattice

\[ \rho^{\mu \nu}(x-y) = \langle [f^{\mu}(x), f^{\nu}(y)] \rangle \]

\[ \rho_{\nu} = -\rho^{00} + \rho^{ii} \]

- \( \rho^{ii} \): electrical conductivity

\[ \sigma_{em} = \frac{1}{2} \frac{\partial}{\partial \omega} \rho^{ii}(\omega) \bigg|_{\omega=0} \]

one loop: \( \rho^{ii}(\omega) = \frac{N_c}{2\pi} \frac{\omega^2}{\omega^2 + 4\pi^2} \left[ 1 - 2\rho(\omega) \right] + \frac{N_c}{2\pi} \frac{\omega^2}{\omega^2 + 4\pi^2} \]

- \( \rho^{00} \): restricted by Ward identities

pinching pole

\[ \sim \frac{\omega^4}{\omega^4 + 4\pi^2} \text{ etc.} \]

\[ \rho^{00}(\omega) = \frac{N_c}{2\pi} \frac{\omega^2}{\omega^2 + 4\pi^2} \text{ for } \omega \gg T \]

\[ \Rightarrow \rho^{00}(\omega) \sim \omega \rho(\omega) \text{ always!} \]

Simple test on respecting Ward Ido in summation schemes.

\[ \rho_{\nu} = -\rho^{00} + \rho^{ii} \]

\[ \rho^{ii}(\omega) \]

\[ \rho^{00}(\omega) \]

Challenge for MEM to disentangle the low-frequency domain

(Karsch et al. find

\( \rho_{\nu}(\omega) \sim 0 \text{ for } \omega \gg T \)
Spectral functions and the classical approximation

- Classical field approximation:
  - Nonperturbative approach to real-time dynamics of soft, highly populated fields
- Spectral functions:
  - Equilibrium QFT
    \[ p(x-y) = i\langle \phi(x,0^+)\phi^+(y,0^-) \rangle \]
    \[ F(x-y) = \frac{i}{2} \langle \phi(x,0^+)\phi^+(y,0^-) \rangle \] symmetrical correlator
  - KMS condition: \[ F(p) = -i[n(p) + \frac{1}{T}] \]

- Classical fields at finite \( T \)
  \[ p_{cl}(x-y) = -\langle \phi(x,0^+)\phi^+(y,0^-) \rangle_{cl} \]
  with Poisson bracket
  \[ \{A(x), B(y)\} = \int d^4z \frac{\delta A(x)}{\delta \phi(z)} \frac{\delta B(y)}{\delta \phi(z)} = \cdots \]
  (difficult to compute)
  \[ S(x-y) = \langle 0\{x,0^+;y,0^-\} \rangle_{cl} \] normal correlator
  classical KMS condition:
  \[ \langle \phi(x) \phi^+(y) \rangle_{cl} \]
  or in
  \[ \delta(x,y) - \frac{1}{T} \Delta_S(x,y) \]

Example: 2+1D scalar field: plasmon

\[ H = \int dx^4 \left( \frac{1}{2} \partial^2 + \frac{1}{2} (\partial_0)^2 + \frac{1}{2} \partial^2 \phi^2 + \frac{1}{4} \phi^4 \right) \]

One-particle spectral function

\[ \rho(p) = i\langle [\phi(x), \phi^+(y)] \rangle \]

\[ \rho_{cl}(p) = 2\pi \delta(p^0 - p^0 - \not{p}) \]

\[ S(x-y) = \langle \phi(x,0^+)\phi^+(y,0^-) \rangle_{cl} \]

KMS:

\[ \rho_{cl}(p) = -\frac{i}{T} \delta(\not{p}) \]

\[ \pi(t,x) = \delta(t,0) \phi(x) \]

Numerical calculation:

- Spatial lattice \( N \times N \), periodic b.c. \( N=128 \)
- Lattice spacing \( a \), \( ma=0.2 \)
- Leapfrog, time step \( a_0 \), \( a_0/a=0.1 \)
- Symmetric definition:
  \[ \rho_{cl}(t,x) = -\frac{i}{T} \langle \pi(t+x,0^+)\pi(t,x) \rangle_{cl} \]

- Temperature \( T = a^2 \langle \pi^2 \rangle \)
- Classical theory: \( \pi \) can be scaled out \( \langle \pi t/N^2 \rangle \) without loss of generality \( T/a^2 \approx 1 \)
- Thermal initial configuration
  - (HMC, Kramers equation)
  - Real-time (Hamiltonian) evolution
  \[ T \approx 2000 \times \]
Spectral function in real time

\[ \rho_{cl}(t, \varphi = 0) \]

temperature \( T/m = 7.2 \)

Spectral function vs frequency

\[ \rho_{cl}(\omega, \varphi = 0) \]

[From sine-transform]

inset: dotted line: Breit-Wigner fit

\[ \rho_{BW}(\omega) = \frac{2\pi \Gamma}{(\omega^2 - M^2)^2 + \omega^2 \Gamma^2} \]
Spectral functions for different temperatures

dotted lines are Breit-Wigner fits

\[ \rho(M) \sim \frac{1}{M^2} \]

\[ \text{Perturbative expectation:} \]

\[ \text{the plasmon} \]

\[ \rho = \frac{-\text{Im} \Sigma_{\text{E}}}{(\omega^2 - m^2 - \text{Re} \Sigma_{\text{E}})^2 + (\text{Im} \Sigma_{\text{E}})^2} \]

\[ \Sigma_{\text{E}} = \text{Re} \Sigma_{\text{E}} + i \text{Im} \Sigma_{\text{E}} \]

\[ \text{retarded self-energy} \]

\[ \Gamma = -\text{Im} \Sigma_{\text{E}} \ll \sqrt{\omega^2 + m^2 + \text{Re} \Sigma_{\text{E}}} \]

\[ \text{narrow width} \]

\[ \rightarrow \text{Breit-Wigner spectral function (at zero momentum)} \]

\[ \rho_{\text{BW}}(\omega) = \frac{2\pi \Gamma}{(\omega^2 - m^2)^2 + \omega^2 \Gamma^2} \]

\[ \rightarrow \text{Fig. 8} \]

\[ \text{Perturbation theory (one-loop resummed) (1+1 D)} \]

- \( M^2 : \)

\[ M^2 = \omega^2 + \frac{\Delta}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{T^+}{p^2 - M^2} \]

\[ \rightarrow \text{lattice gap equation for} \ M \]

- \( \Gamma : \)

\[ \Gamma = \frac{\lambda^2 T^+}{M^2} \]

\[ \Delta = \frac{3 - 2 \sqrt{3}}{8 \pi} \approx 0.0017 \]

\[ \text{(new calculation)} \]

\[ \rightarrow \text{Compare with nonperturbative plasmon mass and} \]
Temperature dependence of perturbative and nonperturbative classical plasma mass $M$ and width $\Gamma$.

Data points from RW-fit, error from jackknife.

Perturbation theory is applicable.

Conclusions

- Transport coefficients:
  - Analytical calculation in field theory nontrivial.
- From the lattice:
  - Euclidean correlators insensitive to transport coefficients and to details of soft dynamics ($\omega \ll T$) in general.
- Generic feature:
  - Challenge for Maximal Entropy Method.
  - Example: Thermal dilepton rate.
- Spectral function directly in real time:
  - Classical approximation.
  - Example: Plasma.