Finite baryon density &
gauge field dynamics
M. Laine (CERN)

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Take QCD with $N_f$ flavours, temperature $T$, quark chemical potential $\mu$, with $\max(\mu, T) \gg \Lambda_{\text{QCD}}$.

Integrate out perturbative modes, $P \geq \max(\mu, T)$. What is effective theory for modes with $P \leq \max(gT, g\mu)$?

(If perturbative, eff. theory $\equiv$ quasiparticle picture.
If not, still valid but needs lattice simulations.)

Static observables ($p_\perp \to 0$, $q_\perp \leq qT$) $\Rightarrow$
Dimensionally Reduced (DR) effective action.

Non-static observables ($p_\perp$, $q_\perp \leq \max(gT, g\mu)$) $\Rightarrow$
Hard Thermal/Dense Loop (HTL/HDL) action.

$\mathcal{S}_{\text{DR}} = m_0^2 \text{Tr} \left( A_0^2 + \ldots \right)$,

$m_0^2 = \frac{N_c}{3} g^2 T^2 + \frac{N_f}{2} g^2 \left( \frac{T^2}{3} + \frac{\mu^2}{T^2} \right) + \ldots$
Effective theories have often extra symmetries, only broken by higher-dimensional operators. 

E.g., strangeness violations in QCD induced by 

\[ \delta Y_E = \frac{g_s^2}{8 \pi^2} \sin 2\theta_c \left( (\bar{u}_L \gamma_\mu u_L) (\bar{u}_L \gamma_\mu s_L) + \ldots \right). \]

In analogy, inclusion of \( \mu \), 

\[ \delta Y_E = \frac{1}{2} \text{Tr} F_{\mu \nu} F^{\mu \nu} + \bar{\psi} \left[ \gamma_\mu D_\mu - \gamma_\mu \mu \right] \psi, \]

breaks charge conjugation invariance, \( \mathcal{C} \). Does this lead to new operators in HTL/HQCD/DR theories? 

Since \( \mathcal{C} \) can be compensated for by \( \mu \rightarrow -\mu \), coefficients must be odd in \( \mu \rightarrow \gamma \)

\[ \delta Y_E = i\mu N_f \frac{g_s^3}{8 \pi^2} \text{Tr} \left( A_0^2 + \ldots \right). \]

**Applications:**

- \( \left< \text{Tr} F_{\mu \nu}^3 (\bar{u}, t) \text{Tr} F_{03} F_{12}^\dagger (0, 0) \right> \neq 0 \)
- \( \left< \text{Tr} \left[ \pi + p^+ \right] (\bar{u}, 0) \text{Tr} \left[ \pi - p^+ \right] (0, 0) \right> \neq 0 \)  
  \( \text{Arnold, Yaffe} \)

- \( \left< n_B (\bar{u}, t) e (0, 0) \right> \neq 0 \)  
  \( \text{Bromoff, Korthals Altes, Kharzeev, Pisarski, Tjon} \)

- \( \sum \frac{1 \pi + 2 - \pi - 2}{1 \pi + 2 + \pi - 2} \neq 0 \)  
- \( M \)

- Off-diagonal quark number susceptibility 
  \( \left< \sum_{\bar{u} \gamma_\nu u} (\bar{u}, 0) \bar{\gamma}_\nu u (0, 0) \right> \)  
  \( \text{Blaizot, Iancu, Rebhan} \)

- The "sign problem" in the static limit.

**Environments:**

- In cosmology \( \frac{\mu}{\lambda} \sim 10^{-8} \).
- \( V \) At and above AGS & SPS energies, \( \frac{\mu}{\lambda} \leq 1.3. \)
- \( V \) Fodor & Katz \( \Rightarrow \left( \frac{\mu}{\lambda} \right) \text{tricritical} \sim 1.5. \)
- In neutron stars \( \frac{\mu}{\lambda} \sim \infty. \)
Leading order contributions

\[ A_{\mu}(Q_0) \]

\[ \Lambda_{\mu}(Q_{\mu}) \]

Assume \( Q_0 \leq \text{max} (gT, g\mu) \). Do first Matsubara sum, expand then in \( \frac{Q_0}{1\beta} \); phase space integral \( S\bar{F}C \ldots \) contains \( \frac{1}{e^{\beta(q^0 + \mu)} + 1} \) and is dominated by \( q^0 = \text{max} (T, \mu) \).

\[ n = 1 \rightarrow S \bar{X}_F = -igN_f \frac{N_c}{3} (T^2 + \frac{\mu^2}{3}) \text{Tr} A_0 = 0. \]

\[ n = 2 \rightarrow \text{HTL / HDL of Braaten - Pisarski}. \]

\[ n = 3 \rightarrow S \bar{X}_M = -\frac{1}{2\pi^2} g^3 \mu N_f \int_v \text{Tr} [\tilde{\Lambda}_\mu \tilde{\Lambda}_\nu \frac{1}{v \cdot D} \delta^\nu_a \tilde{\Lambda}_a], \]

\[ \tilde{\Lambda}_\mu = (\delta_\mu - \frac{v^\mu \partial_\mu}{v \cdot D}) A_\mu, \]

\[ v^\mu = (1, v^i), \quad v \cdot v = 0. \]

This result is not gauge invariant; for "soft" fields \( (\omega g\Lambda = g^2 T) \) higher \( n \) contribute at same order.

Fortunately, graphs are easily analysed with gauge choice \( v \cdot \Lambda = 0 \) [Frenkel, Taylor; Elmfors et al].

Turns out only \( n=4 \) could give a contribution \( \propto \mu \), but explicit computation shows result vanishes.

Thus, can remove gauge choice by writing previous result in a gauge invariant way:

\[ \tilde{\Lambda}_\mu = \frac{1}{v \cdot D} v^\mu F_{\mu \nu}, \ldots \Rightarrow \]

\[ S \bar{X}_M = -\frac{1}{2\pi^2} g^3 \mu N_f \int_v \text{Tr} \left( \frac{1}{v \cdot D} v^\mu F_{\mu \nu} \right) \left( \frac{1}{v \cdot D} v^\nu F_{\nu \rho} \right) \times \left( \frac{1}{v \cdot D} D^3 \frac{1}{v \cdot D} v^\nu F_{\nu \rho} \right). \]
Fortunately, result can be written in a simpler, local form, by introducing additional d.o.f.'s, as suggested by classical kinetic theory!

\[ f^{(c)}[A_\mu(x); x, \lambda] \equiv N_c \times N_c - \text{matrices} \]

\[ f^{(c)}[0; x, \lambda] \equiv \begin{cases} \frac{\lambda_i}{N_c} \frac{2(\bar{p}) \bar{p} \bar{p}}{e^{\mu^2 \bar{p}^2} + 1} ; & i = 1, \ldots, 2N_c N_f \\ (\mu \neq \mu) ; & i = 2N_c N_f + 1, \ldots, 4N_c N_f \end{cases} \]

\[ \left\{ \begin{align*} [p \cdot D, f^{(c)}] + \frac{g_i}{2} \left\{ p^\mu F^\mu_{\nu}, \frac{\delta f^{(c)}}{\delta p_\nu} \right\} &= 0 \\ \delta \left[ S^{(c)}_A \right] &= \sum g_i \int \frac{d^3 p}{(2\pi)^3} p^\mu \text{Tr} \left[ T^a f^{(c)} \right] \end{align*} \] \]

This system can in principle be put on a 4+3-dimensional lattice and studied numerically, as has been done in the HTL-case.

[C.D. Moore et al.; D. Bodeker et al.; Rajantie & Hindmarsh]

Let us study in more detail the static case. Then system is local even without \( f^{(c)} \):

\[ \mathcal{L}_E = \frac{1}{2} \text{Tr} F_{ij}^2 + \text{Tr} \left[ \left( D_i \lambda_\alpha \lambda_\beta \right)^2 \right] \\
+ g_s \left[ T^a \left( \frac{N_c}{3} + N_f \right) + \frac{u^2}{2 \pi c} \right] \text{Tr} A^2 + ig^2 \mu \frac{N_f}{3 \pi^2} \text{Tr} A^3 \\
+ g^4 \frac{6 + N_c - N_f}{2 \pi^2} \left( \text{Tr} A^2 \right)^2 \]

\[ \text{Tr} e^{-\beta H} \Omega(x, 0) \Omega(x, 0) \ldots \Rightarrow \int \text{DA}_x e^{-\beta S_x} \hat{\Omega}(x, 0) \ldots \]

Starting directly from Matsubara formalism, NLO corrections to action have also been computed.

Observables:

1. \( \sim \text{Tr} F_{01} F_{12} \rightarrow \) colour-electric screening
2. \( \text{Tr} F_{01} F_{12} \rightarrow \) colour-magnetic susceptibilities
3. \( \lambda_\lambda \psi \) free energy

Eff. theory reliable down to \( T \sim 2T_c \).
The C-odd operator causes now a "sign-problem":
\[ \int \mathcal{D}A_\mu \mathcal{D}E e^{-S_{E}\left(p\right)} \sim \int \mathcal{D}A_\mu \mathcal{D}E e^{-S_{E}\left(0\right)} e^{-i \frac{\mu}{\hbar} \int \frac{g^3}{8} \text{Tr} A_0^3} \]

How serious is it?

- \[ \langle \left[ g^3 \frac{\text{Tr} A_0^2}{8} \right] \rangle_{\mu=0} = 0 \]
- \[ \text{width} = \left\{ \langle \left[ \ldots I^2 \right] \rangle \right\}^{1/2} = \quad \bigcirc \]

\[ = \frac{g^3 N_f}{12 \pi^3} \left[ 5 V T^3 \left( \ln \frac{1}{a m_0} + \# \right) \right]^{1/2} \]

Turns out that for \( \frac{\mu}{T} \leq 4.0 \) and realistic \( V \), this is not too serious at all!

Numerical distribution of \( \frac{1}{2} g^3 \frac{\text{Tr} A_0^2}{8} \), obtained by doing importance sampling with the real part of the action:

![Numerical distribution graphs]

Legend:
- \( \mu/T=1.0 \)
- \( \mu/T=2.5 \)
- \( \mu/T=3.0 \)
- \( \mu/T=4.0 \)
Measurements of correlation lengths with the effective theory can now be used to test various methods for simulating $\mu \neq 0$.

1. "Reweighting":

\[ \int D\alpha_\mu \left[ \sigma \right] e^{-R \sigma_\mu - i\lambda \sigma_\mu} \rightarrow \int D\alpha_\mu \left[ \sigma \right] e^{-i\lambda \sigma_\mu} e^{-R \sigma_\mu} \]

Because of the narrow distribution, reliable infinite $V$ estimates can be obtained up to $\mu \leq \pi T$!

An analogous method has been used in 4d simulations by Fodor & Katz, to study the phase diagram. However, generically oscillations become more serious as $-V$, so in this case eff. theory may be qualitatively better with $-V^{1/2}$.

2. "Imaginary chemical potential".

Consider $C$-even observable analytic in $\mu$. Then $\langle 0 \rangle = c_0 + c_1 (\frac{1}{\pi})^2 + c_2 (\frac{\mu}{\pi})^3 + \ldots$

Measure $c_i$ by using imaginary $\mu$, whereby action is real. Then test against reweighting:

\[ \text{Tr } F_{\alpha \beta} \; \langle \text{Tr } F_{\alpha \beta} \rangle \quad \text{Tr } F_{\alpha \beta} F_{\gamma \delta} \]

Success not specific to eff. theory. Recently used in 4d simulations to study phase diagram by de Forcrand $\&$ Philipsen.
3. "Taylor expansions"

The coefficients $c_i$ could be measured from operators at $\mu = 0$!
Orders of magnitude:

\[
\begin{array}{ccc}
0^+ & 4.0 & 3.0 & -1.0 \\
0^- & 5.8 & 2.0 & -1.0 \\
\end{array}
\]

Again, nothing is specific to eff. theory. For $\neq \pi$, $c_0, c_1$ suffice.
Used in 4d simulations to study phase diagram by Allton et al.

Summary: static case & sign problem

- DR theory allows to put aside real problem of 4d lattice, chiral quarks, and concentrate on $\text{Im} S_F$.
- Expansion parameter $\sim (\frac{\text{H}}{\rho})^2 \Rightarrow$ in the RHIC regime, many methods work and are $\sim$ equivalent.
- For reweighting to work, it is very helpful if oscillations set in as $\sim \xi \sqrt{\tau}$, not as $\sim \xi$.
- Correlation lengths non-perturbative but known with great precision — however how to connect to $\Phi$?
Outlook: non-static case

- For $T \geq 2T_c$, eff. theory allows in principle to determine oscillation frequencies, damping rates, etc, non-perturbatively — even at finite $\mu$!

- However a NLO derivation, as well as an exhaustive analysis of UV divergences and renormalisation, are still missing.